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PRELIMINARY REPORT ON HOMOLOGICAL ALGEBRA.

Comments of the writer. This report is based on the forth-coming book by Cartan and Eilenberg entitled "Homological Algebra" (referred to as HA). The arrangement of the material is however considerably different than in HA.

This being a 0th approximation of a plan, everything was omitted that was not essential for the basic plan. The items sacrificed include

(a) The general theory of functors, satellited and derided functors.

(b) The multiplicative theory. This omission is of course purely temporary as the multiplicative theory is essential and will have to be reinstated. However, in the writers opinion, the basic plan can be seen clearer without bringing in the multiplicative theory right away.

(c) The cohomology theory of algebras à la Hochschild. It is not clear yet whether this theory is essential for Bourbaki.

The report is devided into three chapters. The first one is purely formal and is intended to include everything needed in topological homology theory, except the Kunneth relations. The second part deals with the functors Ext and Tor and includes the Kunneth relations. The third one is very sketchy and is intended to show how the homology and cohomology theories of groups and Lie algebras fit into the scheme of Ch.II. A separate report concerning these questions will be needed.

All the missing detailed definitions, statements and proofs will be found in HA. Chapter XIV of HA dealing with spectral sequences is included as part of this report.

Chapter I.

§ 1. Exact sequences.

In this section exact sequences are introduced and various formal properties are established. Most of these properties express themselves by means of diagrams.

Example 1 (the five lemma). Given a commutative diagram

$$\begin{array}{ccccccccc} A_2 & \longrightarrow & A_1 & \longrightarrow & A_o & \longrightarrow & A_{-1} & \longrightarrow & A_{-2} \\ \downarrow h_2 & & \downarrow h_1 & & \downarrow h_o & & \downarrow h_{-1} & & \downarrow h_{-2} \\ B_2 & \longrightarrow & B_1 & \longrightarrow & B_o & \longrightarrow & B_{-1} & \longrightarrow & B_{-2} \end{array}$$

with exact rows, then

(a) $\operatorname{Coker} h_2 = 0$, $\operatorname{Ker} h_1 = 0$, $\operatorname{Ker} h_{-1} = 0 \Longrightarrow \operatorname{Ker} h_o = 0$

(b) $\operatorname{Ker} h_{-2} = 0$, $\operatorname{Coker} h_{-1} = 0$, $\operatorname{Coker} h_1 = 0 \Longrightarrow \operatorname{Coker} h_o = 0$.

Corollary : If h_1 and h_{-1} are isomorphisms, h_2 is an epimorphism and h_{-2} is a monomorphism then h_o if an isomorphism.

Example 2. Consider a commutative diagram

$$\begin{array}{ccccccccc} & & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & & \end{array}$$

with exact rows. One then defines a homomorphism

$$\Delta : \operatorname{Ker} f'' \longrightarrow \operatorname{Coker} f'$$

and proves that the sequence

$$\operatorname{Ker} f' \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \xrightarrow{\Delta} \operatorname{Coker} f' \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} f'$$

is exact.

This section could also include the discussion of the behavior of exact sequences vis à vis direct sums, direct products, injective limits and projective limits.

§ 2. Graded modules.

A graded module A is a direct sum $\sum A^i$ of modules $i = 0, \pm 1, \pm 2, \ldots$. Convention $A_i = A^{-i}$. A map $f : A \to C$ of graded modules has degree t if $f(A^i) \subset C^{i+t}$.

A n-tuple graded module A is a direct sum $\sum A^{i_1, \ldots, i_n}$ where $i_1, \ldots, i_n$ are integers. An n-tuple graded module defines a graded module by setting

$$A^i = \sum A^{i_1, \ldots, i_n} \qquad i_1 + \ldots + i_n = i$$

(in pratice, this summation is usually finite).

A map $f : A \to C$ of n-tuple graded modules has degree $(t_1, \ldots, t_n)$ if $f(A^{i_1, \ldots i_n}) \subset C^{i_1+t_1, \ldots, i_n+t_n}$. The integer $t = t_1 + \ldots + t_n$ is called the total degree of f .

A module A (non-graded) may be regarded as an n-tuple graded module by setting $A = A^{0, \ldots, 0}$. A singly graded module A may be regarded as a doubly graded module in two ways : $A^{i,0} = A^i$ or $A^{0,i} = A^i$.

The notion of a graded module can be generalized. Instead of using integers or sequences of integers as degrees, we can assume that the degrees belong to a commutative group G . For later purposes the group G must be given together with a homomorphism $\pi : G \to Z_2$ called the parity function.

§ 3. Modules with differentiation.

A differentiation d in a module A is an endomorphism satisfying dd = 0. Introduce notation

$$Z(A) = \text{Ker } d \quad , \quad Z'(A) = \text{Coker } d$$

$$B(A) = \text{Im } d \quad , \quad B'(A) = \text{Coim } d \quad .$$

N-B : We have $B(A) \approx B'(A)$ but the identification should not be made.

The differentiation d admits a factorization

$$A \to Z'(A) \to B'(A) \to B(A) \to Z(A) \to A .$$

From this diagram we obtain the maps

$$0 \to B(A) \to Z(A) \to Z'(A) \to B'(A) \to 0$$

which form an exact sequence. Define homology module

$$H(A) = Z(A)/B(A) = \mathrm{Ker}(Z'(A) \to B'(A)) .$$

There results exact sequence

$$0 \to H(A) \to Z'(A) \xrightarrow{\tilde{d}} Z(A) \to H(A) \to 0$$

where $\tilde{d}$ is induced by d .

Let A and C be modules with differentiation (both denoted by d). A map $f : A \to C$ is a module homomorphism such that $df = fd$. A map f induces maps $Z(f) : Z(A) \to Z(C), \dots, H(f): H(A) \to H(C)$.

Let f,g be two maps $A \to C$. A homotopy $s : f \simeq g$ is a homomorphism $s : A \to C$ such that $ds + sd = g - f$. If f and g are homotopic then $H(f) = H(g)$.

A module A without differentiation may be regarded as a module with differentiation zero. Then $Z(A) = Z'(A) = H(A) = A$, $B(A) = B'(A) = 0$. If A is a module with differentiation then $Z(A)$, $B(A)$, $Z'(A)$, $B'(A)$, $H(A)$ are modules with zero differentiation.

Let

$$0 \to A' \to A \to A'' \to 0$$

be an exact sequence of modules with differentiation. We consider the commutative diagram

$$\begin{array}{ccccccc}
 & & Z'(A') & \to & Z'(A) & \to & Z'(A'') & \to & 0 \\
 & & \downarrow \tilde{d} & & \downarrow \tilde{d} & & \downarrow \tilde{d} & & \\
0 & \to & Z(A') & \to & Z(A) & \to & Z(A'') & &
\end{array}$$

and prove that the rows are exact. Now apply Example 2 above. There results a map

$$\Delta : H(A'') \longrightarrow H(A')$$

called the <u>connecting</u> homomorphism and the sequence

$$\cdots \longrightarrow H(A') \longrightarrow H(A) \longrightarrow H(A'') \xrightarrow{\Delta} H(A') \longrightarrow H(A) \longrightarrow H(A'') \longrightarrow \cdots$$

is exact. This is the <u>homology sequence</u>.

If

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows, then the diagram

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & H(A') & \longrightarrow & H(A) & \longrightarrow & H(A'') & \longrightarrow & H(A') & \longrightarrow & H(A) & \longrightarrow & H(A'') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H(C') & \longrightarrow & H(C) & \longrightarrow & H(C'') & \longrightarrow & H(C') & \longrightarrow & H(C) & \longrightarrow & H(C'') & \longrightarrow & \cdots \end{array}$$

is commutative.

Behavior of above operators with respect to sums, products, and limits.

§ 4. <u>Complexes</u>.

A <u>complex</u> is a graded module with a differentiation d of degree $+1$. Then $Z(A),\ldots,H(A)$ are graded modules. The natural isomorphism $B'(A) \approx B(A)$ has degree $+1$ and this is the reason for avoiding the identification.

Maps $f : A \longrightarrow C$ of complexes will always be assumed to be of degree 0. Homotopies have degree -1. The homology sequence takes the form

$$\cdots \longrightarrow H^n(A') \longrightarrow H^n(A) \longrightarrow H^n(A'') \xrightarrow{\Delta} H^{n+1}(A') \longrightarrow H^{n+1}(A) \longrightarrow \cdots .$$

An n-tuple complex A is an n-tuple graded module with endomorphisms $d_1,\ldots,d_n$ such that

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(1°) d_i has degree +1 with respect to the index i and has degree zero with respect to other indices.

(2°) $d_i d_i = 0$

(3°) $d_i d_j + d_j d_i = 0$.

It follows that the associated graded module is a complex with the total differentiation $d = d_1 + \dots + d_n$.

Maps $f : A \to C$ of n-tuple complexes are assumed to be of degree $(0,\dots,0)$. If f and g are two such maps we define a homotopy $(s_1,\dots,s_n)$: $f \simeq g$ to be a sequence of homomorphisme $s_i : A \to C$ $i=1,\dots,n$ such that

(1°) s_i has degree -1 with respect to the index i and has degree zero with repsect to other indices.

(2°) $d_i s_j + s_j d_i = 0$ for $i \neq j$

(3°) $\sum d_i s_i + s_i d_i = f-g$.

It follows that $s = s_1 + \dots + s_n$ satisfies $ds + sd = f-g$, so that we obtain a homotopy in the associated complexes.

If A is an n-tuple complex, then we use the symbol H(A) to denote the homology module of A relative to the total differentiation operator d . In addition we may consider the homology module $H_{(i)}(A)$ relative to the differentiation operator d_i . The module $H_{(i)}(A)$ is n-tuply graded and further $d_1,\dots,d_n$ induce in $H_{(i)}(A)$ the structure of an n-tuple complex with i-th differentiation operator zero.

In particular if A is a double complex, then we have the graded module H(A) and the double graded modules $H_{(2)}H_{(1)}(A)$ and $H_{(1)}H_{(2)}(A)$.

§ 5. Filtrations and spectral sequences.

This question has been treated at lenght in Ch.XIV of HA. This chapter is attached as part of this report.

§ 6. Hom and $\otimes$.

Let A and C be left Λ-modules. We shall write $\mathrm{Hom}_\Lambda(A,C)$ for the group of Λ-homomorphisms $A \to C$. If A is a right Λ-module and C is a left Λ-module we consider the tensor products $A \otimes_\Lambda C$, which is an abelian group.

Induced homomorphisms, i.e. Hom_Λ and $\otimes_\Lambda$ are functors.

Behavior with respect to sums, products and limits.

Behavior of Hom_Λ and $\otimes_\Lambda$ with respect exact sequences :

If $A' \to A \to A'' \to 0$ and $0 \to C' \to C \to C''$ are exact then

$$0 \to \mathrm{Hom}(A'',C) \to \mathrm{Hom}(A,C) \to \mathrm{Hom}(A',C)$$

$$0 \to \mathrm{Hom}(A,C') \to \mathrm{Hom}(A,C) \to \mathrm{Hom}(A,C'')$$

$$0 \to \mathrm{Hom}(A'',C') \to \mathrm{Hom}(A,C) \to \mathrm{Hom}(A',C) + \mathrm{Hom}(A,C'')$$

are exact.

If $A' \to A \to A'' \to 0$ and $C' \to C \to C'' \to 0$ are exact then

$$A' \otimes C \to A \otimes C \to A'' \otimes C \to 0$$

$$A \otimes C' \to A \otimes C \to A \otimes C'' \to 0$$

$$A' \otimes C \to A \otimes C' \to A \otimes C \to A'' \otimes C'' \to 0$$

are exact.

§ 7. Graded modules.

Let A and C be graded modules. We consider the doubly graded groups

$$D^{p,q} = \mathrm{Hom}_\Lambda(A_p, C^q) = \mathrm{Hom}_\Lambda(A^{-p}, C^q)$$

(note the change of sign on the contravariant variable). This doubly graded group is denoted by $\mathrm{Hom}_\Lambda(A,C)$, but is <u>not</u> the group of all homomorphisms $A \to C$. The associated graded ~~module~~ group also is denoted by $\mathrm{Hom}_\Lambda(A,C)$.

Let $f: A' \to A$, $g: C \to C'$ be maps of degree r and s respectively. We define the map

$$\mathrm{Hom}(f,g) : \mathrm{Hom}(A,C) \to \mathrm{Hom}(A',C')$$

of degree (r,s) as follows : If $\varphi \in D^{p,q}$ ie. $\varphi : A_p \to C^q$ then

$\mathrm{Hom}(f,g)\varphi = (-1)^{ps} g^q \varphi f_{p+r} : A'_{p+r} \to C'^{q+s}$.

This rule is easier understood on the tensor product. If $f: A \to A'$, $g: C \to C'$ are maps of degree r and s respectively, then $f \otimes g : A \otimes C \to A' \otimes C'$ is a map of degree (r,s) given by

$$(f \otimes g)(a^p \otimes c^q) = (-1)^{ps} fa^p \otimes gc^q$$

(note that a^p and g have changed places !).

If A and V are complexes then $\mathrm{Hom}_\Lambda(A,C)$ is a double complex with differentiations

$$d_1 = \mathrm{Hom}_\Lambda(d_A,C) , \quad d_2 = \mathrm{Hom}_\Lambda(A,d_C) .$$

Noté that these differentiations anti-commute as required. Further, the total differentiation is the usual one (sign included). Similarly $A \otimes C$ is a double complex with

$$d_1 = d_A \otimes C , \quad d_2 = A \otimes d_C .$$

The total differentiation $d = d_1 + d_2$ satisfies

$$d(a^p \otimes C^q) = (da^p) \otimes c^q + (-1)^p a^p \otimes dc^q$$

as usual.

The above needs to be generalized : If A is an k-tuple complex and C is a 1-tuple complex then Hom(A,C) and $A \otimes C$ are (k+1)-tuple complexes.

§ 8. Homomorphismes α and α' .

Let A and C be complexes. Define homomorphisms

$$\alpha' : H(\mathrm{Hom}_\Lambda(A,C)) \longrightarrow \mathrm{Hom}_\Lambda(H(A),H(C))$$

$$\alpha : H(A) \otimes_\Lambda H(C) \longrightarrow H(A \otimes_\Lambda C) .$$

These homomorphisms have the following basic properties

(1°) α and α' are natural relative to maps of A and C .

(2°) α and α' are identities if A and C have zero differentiations.

These two properties characterize α and α' completely. Further important properties are commutation rules with connecting homomorphisms . For instance, if $0 \to C' \to C \to C'' \to 0$ is an exact sequence.

Chapter II.

§ 1. Projective and injective modules.

A module A is *projective* if given any homomorphism $f: A \to B''$ and any epimorphism $g: B \to B''$ there is a homomorphism $h: A \to B$ with $f = g \circ h$.

Equivalent properties:

(1°) If $g: B \to B''$ is an epimorphism them $\mathrm{Hom}_\Lambda(A,g): \mathrm{Hom}_\Lambda(A,B) \to$ $\to \mathrm{Hom}(AB'')$ also is an epimorphism.

(2°) If $0 \to C' \to C \to C'' \to 0$ is exact then $0 \to \mathrm{Hom}(A,C') \to \mathrm{Hom}(A,C) \to \mathrm{Hom}(A,C'') \to 0$ also is exact.

(3°) If $f: \bar{A} \to A$ is an epimorphism the Ker f is a direct summand of $\bar{A}$.

(4°) A is a direct summand of a Λ-free module.

A direct sum of modules is projective if and only if each member is projective.

There is a dual notion of an *injective* module. A module A is injective if given any homomorphism $f: B' \to A$ and any monomorphism $g: B' \to B$ there is a homomorphism $h: B \to A$ such that $f = hg$.

Equivalent properties:

(1°) If $g: B' \to B$ is a monomorphism then $\mathrm{Hom}_\Lambda(g,A): \mathrm{Hom}_\Lambda(B,A) \to$ $\to \mathrm{Hom}_\Lambda(B'A)$ is an epimorphism

(2°) If $0 \to B' \to B \to B'' \to 0$ is exact then $0 \to \mathrm{Hom}(B'',A) \to \mathrm{Hom}(B,A) \to \mathrm{Hom}(B',A) \to 0$ also is exact.

(3°) If $f: A \to \bar{A}$ is a monomorphism then Im f is a direct summand of $\bar{A}$.

(4°) Given a (left) ideal I of Λ and a homomorphism $f: I \to A$ there is an element $a \in A$ such that $fi = ia$ for all $i \in I$.

The topological analogues of injective modules are the absolute retracts.

Basic property : every module is a quotient of a projective module and is a submodule of an injective module.

If A is a projective module and C is a complex then

$$\alpha' : H(\mathrm{Hom}(A,C)) \approx \mathrm{Hom}(A,H(C))$$

and similarly for α and $\otimes_\Lambda$.

If C is an injective module and A is a complex then

$$\alpha' : H(\mathrm{Hom}(A,C)) \approx \mathrm{Hom}(H(A),C) .$$

§ 2. Resolutions.

A projective resolution X of A is a complex

$$\cdots \to X_n \to \cdots \to X_0 \to 0 \to 0 \to \cdots$$

composed of projective modules and a map

$$\begin{array}{ccccccccccc} \to & X_n & \to & \cdots & \to & X_1 & \to & X_0 & \to & 0 & \to 0 \\ & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle\varepsilon} & & & \\ \to & 0 & \to & \cdots & \to & 0 & \to & A & \to & 0 & \end{array}$$

called the augmentation, such that ε induces an isomorphism $H(X) \approx H(A) = A$. This is equivalent with the exactness of the sequence

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \xrightarrow{\varepsilon} A \to 0 \to \cdots$$

Injective resolutions are defined similarly : reverse all arrows and raise all indices.

There is a whole body of propositions concerning projective and injective resolutions (see HA Ch.V). The main ones are : projective resolutions exist for any module and two projective resolutions of the same module have the same homotopy type (i.e. there exist maps $\varphi: X \to X'$ and $\psi: X' \to X$ such that the composition $\psi\varphi$ and $\varphi\psi$ are homotopic to the identity maps. Similarly for injective resolutions.

§ 3. Ext and Tor.

Let A and C be Λ-modules, let X be a projective resolution of A and Y and injective resolution of C. Then $Hom_\Lambda(X,Y)$ is a double complex whose homology module is (up to canonical isomorphisms) independent of the choice of X and Y. We define

$$Ext_\Lambda(A,C) = H(Hom_\Lambda(X,Y))$$

This is a graded module. In addition to the usual functorial properties (contravariant in A and covariant in C) we define connecting homomorphisms : for each exact sequence $0 \to A' \to A \to A'' \to 0$ we have a homomorphism $Ext_\Lambda(A',C) \to Ext_\Lambda(A'',C)$ of degree +1 ; for each exact sequence $0 \to C' \to C \to C'' \to 0$ we have a homomorphism $Ext_\Lambda(A,C'') \to Ext(A,C')$ of degree 1. The following are the main properties of ext.

(1) For each exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence

$$\dots \to Ext^p_\Lambda(A'',C) \to Ext^p(A,C) \to Ext^p(A',C) \to Ext^{p+1}(A'',C) \to \dots$$

is exact.

(1a) For each exact sequence $0 \to C' \to C \to C'' \to 0$ the sequence

$$\dots \to Ext^p(A,C') \to Ext^p(A,C) \to Ext^p(A,C'') \to Ext^p(A,C') \to \dots$$

is exact.

(2) $Ext^p_\Lambda(A,C) = 0$ for $p < 0$, $Ext^0_\Lambda(A,C) = Hom_\Lambda(A,C)$.

(3) $Ext^p_\Lambda(A,C) = 0$ for $p > 0$ and A projective.

(3a) $Ext^p_\Lambda(A,C) = 0$ for $p > 0$ and C injective.

Properties (1), (2) and (3) (or (1a), (2) and (3a)) together with the naturality properties yield an axiomatic description of Ext_Λ .

$\mathrm{Ext}_\Lambda(A,C)$ was defined by taking resolutions of both A and C . It suffices however to resolved only one of the variables. We have

$$H(\mathrm{Hom}_\Lambda(X,C)) \approx \mathrm{Ext}_\Lambda(A,C) \approx H(\mathrm{Hom}_\Lambda(A,Y)) .$$

A similar discussion applies to $A \otimes C$. Here we take projective resolutions of X and Y for A and C . Then

$$\mathrm{Tor}^\Lambda(A,C) = H(X \otimes_\Lambda Y)$$

is essentially independent of X and Y . The properties of Tor are similar to those for Ext.

§ 4. Dimension.

The projective dimension of a module A is $\leq n$ if there is a projective resolution X of A with $X_{n+1} = 0$. The projective dimension of A is zero if and only if A is projective. Injective dimension is defined similarly.

The left global dimension of a ring Λ is the maximum projective dimension of left Λ-modules. The same definition is obtained using injective dimension. Right global dimension is defined using right Λ-modules. The two dimensions coincide in a number of special cases, but the general question is open.

The questions of dimension are intimetely connected with the functors Ext_Λ . Indeed

projective dimension $A \leq n \Longleftrightarrow \mathrm{Ext}_\Lambda^{n+1}(A,C) = 0$ for all C

injective dimension $C \leq n \Longleftrightarrow \mathrm{Ext}_\Lambda^{n+1}(A,C) = 0$ for all A

left global dimension $\leq n \Longleftrightarrow \mathrm{Ext}_\Lambda^{n+1} = 0$.

The following properties are equivalent.

(1) left global dimension $\Lambda = 0$

(2) every left Λ-module is projective

(3) every left Λ-module is injective

(4) every submodule of a left Λ-module is a direct summand

(5) every left Λ-module is a direct sum of simple Λ-modules

(6) every left ideal of Λ is a direct sum of simple Λ-modules.

These rings are called "semi-simple", and have a structure theorem showing that the right global dimension also is zero.

The following properties are equivalent.

(7) left global dimension $\Lambda \leqq 1$

(8) every submodule of a projective left Λ-module is projective

(9) every quotient module of an injective left Λ-module is injective

(10) every left ideal of Λ is projective.

Such rings are called <u>hereditary</u>.

For integral domains "hereditary" = Dedekind .

§ 5. <u>Künneth relations</u>.

A projective (injective) resolution of a module A is a complex . It is natural to expect that the projective (injective) resolution of a complex should be a double complex subjected to various conditions. For a definition see HA , Ch. XVI.

Let A and C be complexes, X a projective resolution of A and Y an injective resolution of C . Then $\mathrm{Hom}_\Lambda(X,Y)$ is a quadruple complex . We regard it as a double complex by grouping the first index with the third and second with the fourth. All the invariants of this double complex are idependent of the choice of X and Y . The terms E_2 for the two spectral sequences can be computed and yield

(1) $$H^p(\mathrm{Ext}^q_\Lambda(A,C)) \underset{p}{\Longrightarrow} H^n(\mathrm{Hom}_\Lambda(X,Y))$$

(2) $$\sum_{p_1+p_2=p} \mathrm{Ext}^q_\Lambda(H_{p_1}(A), H^{p_2}(C)) \underset{q}{\Longrightarrow} H^n(\mathrm{Hom}_\Lambda(X,Y))$$

These are the most general Künneth relations. If we assume that in (1) all terms with $q > 0$ vanish (e.g. if A is Λ-projective or C is Λ-injective) then the sequence (1) collapses and (2) becomes

(3) $$\sum_{p_1+p_2=p} \mathrm{Ext}^q_\Lambda(H_{p_1}(A), H^{p_2}(C)) \underset{q}{\Longrightarrow} H^n(\mathrm{Hom}_\Lambda(A,C))$$

If further Λ is hereditary, then in (3) all terms with $q > 1$ are zero and the spectral sequence (3) reduces to the exact sequence

(4) $$0 \to \mathrm{Ext}^1(H(A),H(C)) \to H(\mathrm{Hom}_\Lambda(A,C)) \xrightarrow{\alpha'} \mathrm{Hom}_\Lambda(H(A),H(C)) \to 0.$$

A similar discussion applies to $A \otimes_\Lambda C$ using projective resolutions X and Y of the complexes A and C . We obtain

(1') $$H_p(\mathrm{Tor}^\Lambda_q(A,C)) \underset{p}{\Longrightarrow} H_n(X \otimes_\Lambda Y)$$

(2') $$\sum_{p_1+p_2=p} \mathrm{Tor}^\Lambda_q(H_{p_1}(A), H_{p_2}(C)) \underset{q}{\Longrightarrow} H_n(X \otimes_\Lambda Y) .$$

If terms in (1') with $q > 0$ are zero then (2') becomes

(3') $$\sum_{p_1+p_2=p} \mathrm{Tor}^\Lambda_q(H_{p_1}(A), H_{p_2}(C)) \underset{q}{\Longrightarrow} H_n(A \otimes_\Lambda C)$$

If Λ is hereditary (3') reduces to the exact sequence

(4') $$0 \to H(A) \otimes_\Lambda H(C) \xrightarrow{\alpha} H(A \otimes_\Lambda C) \to \mathrm{Tor}_1(H(A),H(C)) \to 0 .$$

Chapter III.

§1. K-algebras.

In this part we assume that the ring Λ is a K-algebra where K is a commutative ring i.e. that a ring homomorphism $\eta : K \to \Lambda$ is given such that $\eta(K) \subset$ Center Λ .

A complemented algebra is a K-algebra Λ which is projective as a K-module and for which a ring homomorphism $\varepsilon : \Lambda \longrightarrow K$ is given such that the composition $K \xrightarrow{\eta} \Lambda \xrightarrow{\varepsilon} K$ is the identity. The kernel of ε is a two sided ideal I (called the augmentation ideal or the complementary ideal) and as a K-module Λ is the direct sum I + K . The map ε is called the augmentation and induces in K the structure of a left (and right) Λ-module.

The homology and cohomology groups of Λ are defined as

$$H_n(\Lambda, A) = \mathrm{Tor}_n^{\Lambda}(A, K)$$
$$H^n(\Lambda, C) = \mathrm{Ext}_{\Lambda}^n(K, C)$$

where A is a right Λ-module and C is a left Λ-module. If X is a Λ-projective resolution of K as a left Λ-module then

$$H_n(\Lambda, A) = H_n(A \otimes_{\Lambda} X)$$

$$H^n(\Lambda, C) = H^n(\mathrm{Hom}_{\Lambda}(X, C))$$

For various algebras Λ "nice" complexes X may be chosen. One such complex is the standard complex $S(\Lambda)$ which can be constructed for any complemented ring Λ . We define

$$S_n(\Lambda) = \Lambda \otimes \underbrace{I \otimes \dots \otimes I}_{n\text{-times}} \qquad (\otimes = \otimes_K)$$

A typical element of $S_n(\Lambda)$ will be written as

$$\lambda[i_1,\dots,i_n] \qquad \lambda \in \Lambda, \quad i_j \in I .$$

The differentiation is defined as

$$d\lambda i_1,\dots,i_n = \lambda d(i_1,\dots,i_n)$$

$$d[i_1,\dots,i_n] = i_1(i_2,\dots,i_n) + \sum_{j=1}^{n-1} (-1)^j (i_1,\dots,i_j i_{j+1},\dots,i_n) .$$

If we denote

$$[\lambda_1,\dots,\lambda_n] = (\lambda_1 - \varepsilon\lambda_1,\dots,\lambda_n - \varepsilon\lambda_n) \qquad \lambda_i \in \Lambda$$

then

$$d[\lambda_1,\dots,\lambda_n] = \lambda_1[\lambda_2,\dots,\lambda_n]$$
$$+ \sum (-1)^j [\lambda_1,\dots,\lambda_j\lambda_{j+1},\dots,\lambda_n]$$
$$+ (-1)^n [\lambda_1,\dots,\lambda_{n-1}](\varepsilon\lambda_n) .$$

§ 2. Groups.

Let Π be a (multiplicative) group. We define $\Lambda = K(\Pi)$ to be the K-algebra of Π. Then a Λ-module is a K-module on which Π operates as a group of K-endomorphisms. We convert Λ into a complemented K-algebra by setting $\varepsilon x = 1$ for $x \in \Pi$. The homology and cohomology groups of this complemented algebra Λ are then the homology and cohomology groups of Π.

§ 3. Lie algebras.

Let L be a Lie algebra over K which is K-free. Let $\Lambda = E(L)$ be the enveloping algebra of L. Then a left Λ-module is a representation of L and vice versa. Λ is complemented by setting $\varepsilon x = 0$ for $x \in L$. Again the homology and cohomology groups of Λ are the homology and cohomology groups of the Lie algebra L.

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Faudra faire d'abord mettre en évidence ces trucs de suites exactes relatifs aux groupes

II. ABELIAN CATEGORIES

1. Definition.

Let $\mathcal{A}$ be a category such that $H(A,B)$ is an abelian group, for any object $A,B \in \mathcal{A}$. We shall say that $\mathcal{A}$ is an <u>abelian category</u> if the following axioms hold:

(A.1) The composition $H(B,C) \times H(A,B) \to H(A,C)$ is bilinear.

(A.2) For every $u: A \to B$, there exist maps

$$C \xrightarrow{t} A \xrightarrow{v} D \xrightarrow{w} B \xrightarrow{z} E$$

C : noyau
D : image (ou coimage ; on distinguera + tard)
E : conoyau.

such that $u = wv$ and for each $X \in \mathcal{A}$ the following sequences are exact:

$$0 \to H(X,C) \to H(X,A) \to H(X,D)$$

$$0 \to H(X,D) \to H(X,B) \to H(X,E)$$

$$0 \to H(D,X) \to H(A,X) \to H(C,X)$$

$$0 \to H(E,X) \to H(B,X) \to H(D,X).$$

(A.3) Given $A_1, A_2 \in \mathcal{A}$ there exists $A \in \mathcal{A}$ and maps

$$A_1 \xrightarrow{i_1} A \xrightarrow{p_1} A_1 \qquad A_2 \xrightarrow{i_2} A \xrightarrow{p_2} A_2$$

Somme directe

such that

$$p_1 i_1 = 1_{A_1}, \quad p_2 i_2 = 1_{A_2}, \quad p_2 i_1 = 0 = p_1 i_2, \quad i_1 p_1 + i_2 p_2 = 1_A.$$

The axioms are self dual and thus $\mathcal{A}^*$ also is an abelian category.

It follows from (A.1) that $H(A,A)$ is a ring with 1_A as unit. Each $H(A,B)$ is a right $H(A,A)$-and a left $H(B,B)$- bi-module. The composition is a map

$$H(B,C) \otimes_{H(B,B)} H(A,B) \to H(A,C)$$

and is a left $H(C,C)$- and right $H(A,A)$-homomorphism.

<u>Proposition 1</u>. For any $A \in \mathcal{A}$ the following properties are equivalent:

(1) $1_A = 0$

(2) $H(A,A) = 0$

(3) $H(A,X) = 0$ for all X

(4) $H(X,A) = 0$ for all X.

Proof. Trivial.

An element A having the properties listed in the proposition is called a <u>zero</u> element; notation $A = 0$. If A' is another zero element then the map $0: A \to A'$ is an isomorphism. Thus all zero elements of $\mathcal{A}$ form a class with unique isomorphisms. The existence of zero elements will be proved as a consequence of (A.2).

Proposition 2. A monomorphism $u: A \to B$ is zero if and only if $A = 0$. An epimorphism $v: A \to B$ is zero if and only if $B = 0$.

Proof. If $u = 0$ then $H(X,A) \to H(X,B)$ is zero. Since it also is a monomorphism it follows that $H(X,A) = 0$, i.e., $A = 0$. Conversely, if $A = 0$ then $u = 0$ because $H(A,B) = 0$. Second half is dual.

2. Kernels, images, etc.

We now pass to a discussion of (A.2). We note that t and w are monomorphisms and v and z are epimorphisms. Further since the sequence

$$0 \to H(C,C) \to H(C,A) \to H(D,A)$$

is exact it follows that

$$vt = 0, \qquad ut = 0.$$

Similarly,

$$zw = 0, \qquad zu = 0.$$

Proposition 1. Consider a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[u']{} & B' \end{array}$$

and let

$$C \xrightarrow{t} A \xrightarrow{v} D \xrightarrow{w} B \xrightarrow{z} E$$

$$C' \xrightarrow{t'} A' \xrightarrow{v'} D' \xrightarrow{w'} B' \xrightarrow{z'} E'$$

be maps given by (A.2) for u and u'. Then there exist unique maps

$$c: C \to C', \qquad d: D \to D', \qquad e: E \to E'$$

such that the diagram

$$\begin{array}{ccccccccc} C & \to & A & \to & D & \to & B & \to & E \\ \downarrow && \downarrow && \downarrow && \downarrow && \downarrow \\ C' & \to & A' & \to & D' & \to & B' & \to & E' \end{array}$$

is commutative.

Proof. Since $ut = 0$ we have $0 = but = u'at = w'v'at$. However w' is a monomorphism and thus $v'at = 0$. Since the sequence

$$0 \to H(C,C') \to H(C,A') \to H(C,D')$$

is exact and the element $at \in H(C,A')$ yields zero in $H(C,D')$ there exists a unique $c\colon C \to C'$ such that $t'c = at$. The existence and uniqueness of $e\colon E \to E'$ is dual.

Since $zw = 0$ we have $0 = ezw = z'bw$. Since the sequence

$$0 \to H(D,D') \to H(D,B') \to H(D,E')$$

is exact and the element $bw \in H(D,B')$ yields zero in $H(D,E)$ there exists a unique $d \in H(D,D')$ such that $w'd = bw$. Now we have

$$w'v'a = u'a = bu = bwv = w'dv.$$

Since w' is a monomorphism it follows that $v'a = dv$. The proof is now complete.

Corollary. If in Prop. 1 a and b are isomorphisms, then c, d, e are also isomorphisms.

In particular, applying Prop. and Cor. to the case $A = A'$, $B = B'$, $a = 1_A$, $b = 1_B$ we find that the monomorphism classes $t\colon C \to A$ and $w\colon D \to B$ are uniquely determined by u. Similarly the epimorphism classes $v\colon A \to D$ and $z\colon B \to E$ are uniquely determined. We adopt the following terminology:

$$\begin{aligned} \operatorname{Ker} u &= \text{class of } t, \\ \text{Co-im } u &= \text{class of } v, \\ \operatorname{Im} u &= \text{class of } w, \\ \operatorname{Coker} u &= \text{class of } z. \end{aligned}$$

Proposition 2. The following properties are equivalent:

(1) u is a monomorphism;

(2) $C = 0$ (i.e. Ker $u = 0$);

(3) v is an isomorphism (i.e. Co-im $u = 1_A$);

(4) $u = \text{Im } u$ (i.e. u is an element of the class Im u).

Proof. If (1) holds then $H(X,A) \to H(X,B)$ is a monomorphism. Therefore $H(X,C) = 0$ and $C = 0$. If (2) holds then $H(D,X) \to H(A,X)$ induced by $v: A \to 0$ is an isomorphism. Thus by v is an isomorphism. If (3) holds then we may replace D by A so that $v = 1_A$. Then $w = u$, which is then a monomorphism.

Proposition 2*(dual). The following properties are equivalent:

(1) u is an epimorphism;

(2) $E = 0$ (i.e. Coker $u = 0$);

(3) w is an isomorphism;

(4) $u = \text{Co-im } u$.

Proposition 4. The following properties are equivalent:

Les EVT ain't no abelian category

(1) u is an isomorphism;

(2) u is a monomorphism and an epimorphism;

(3) $C = 0$ and $E = 0$.

Proof. The implication (1) $\Rightarrow$ (2) is clear while the implication (2) $\Rightarrow$ (3) follows from Prop. and Prop. . If $C = 0$ then v is an isomorphism and if $E = 0$ then w is an isomorphism. Thus if $C = 0$ and $E = 0$ then $u = wv$ is an isomorphism.

Corollary. Every (not empty) abelian category contains zero objects.

(Noyau ou conoyau d'une identité)

3. Exact sequences (sketch only).

A sequence

$$(*) \qquad A_i \xrightarrow{u_i} A_{i-1} \xrightarrow{u_{i-1}} \cdots \longrightarrow A_{j+1} \xrightarrow{u_{j+1}} A_j, \qquad j \leq i,$$

is called exact if for every n such that $j < n < i$ we have

$$\text{Ker } u_n = \text{Im } u_{n+1}.$$

The equality is understood as a monomorphism class in $M(A_n)$. Dual and equivalent definition:

ce sont les "sous trucs" de A_n

$$\text{Co-im } u_n = \text{Coker } u_{n+1}.$$

c'est les "quotients" de A_n

Here equality is interpreted as an epimorphism class in $E(A_n)$.

Proposition. A sequence $0 \to A' \to A \to A''$ is exact if and only if

$$0 \to H(X,A') \to H(X,A) \to H(X,A'')$$

is exact for every X.

Proposition. A sequence $A' \to A \to A'' \to 0$ is exact if and only if

$$0 \to H(A'',X) \to H(A,X) \to H(A',X)$$

is exact for every X.

Proposition. A map $u: A' \to A$ is a monomorphism if and only if

$$0 \to A' \xrightarrow{u} A$$

is exact. There exists then an essentially unique epimorphism $v: A \to A''$ such that

$$0 \longrightarrow A' \xrightarrow{u} A \xrightarrow{v} A'' \longrightarrow 0$$

is exact.

Proposition, dual.

Proposition. In the sequence

(A) $$A_i \xrightarrow{u_i} A_{i-1} \longrightarrow \dots \longrightarrow A_j$$

consider for each n, $j \leq n < i$ the factorization

$$A_n \xrightarrow{v_n} Z_{n-1} \xrightarrow{w_{n-1}} A_{n-1}, \qquad u_n = w_{n-1}v_n$$

where v_n is an epimorphism and w_{n-1} is a monomorphism. The sequence (A) is exact if and only if each of the sequences

$$0 \longrightarrow Z_n \xrightarrow{w_n} A_n \xrightarrow{v_n} Z_{n-1} \longrightarrow 0$$

is exact for $j < n < i$.

This last formulation is self-dual.

Proposition. For every map $u: A \to B$ we have the exact sequence

$$0 \to C \xrightarrow{t} A \xrightarrow{u} B \xrightarrow{z} E \to 0$$

given by (A.2).

4. Direct sums, products.

Let $$u_i: A_i \longrightarrow A, \quad i \in I.$$

We shall say that the u_i are a representation of A as a direct sum of A_i if for each X the map

$$H(A,X) \longrightarrow \prod_{i \varepsilon I} H(A_i,X)$$

given by

$$v \longrightarrow \{vu_i\}$$

is an isomorphism.

C'est général, ça appartient au § I

Elementary properties of direct sums.

1) There exists a unique set of maps

$$v_i\colon A \longrightarrow A_i$$

such that

$$v_i u_i = 1_{A_i}, \quad v_i u_j = 0 \text{ for } i \neq j.$$

Indeed, take $X = A_i$ above and consider the element of $\prod_j H(A_j, A_i)$ with i-th coordinate 1_{A_i} and all others zero.

2) Each u_i is a monomorphism and each v_i is an epimorphism.

Indeed, let $w, w'\colon X \to A_i$ and assume $u_i w = u_i w'$. Then

$$w = v_i u_i w = v_i u_i w' = w'.$$

Similarly for v_i.

3) Let

$$u_i\colon A_i \to A, \quad u_i'\colon A_i' \to A'$$

be direct sums. Given any set of maps

$$w_i\colon A_i \to A_i'$$

there exists a unique

$$w\colon A \to A'$$

such that each diagram

$$\begin{array}{ccc} A_i & \xrightarrow{w_i} & A_i' \\ u_i \downarrow & & \downarrow u_i' \\ A & \xrightarrow[w]{} & A' \end{array}$$

is commutative. Then also $v_i' w = w_i v_i$. w is an epimorphism if and only if each w_i is. If w is a monomorphism then each w_i also is a monomorphism.

Dual definition and properties of direct products $v_i\colon A \to A_i$.

Proposition 1. Let I be a finite set and consider maps

$$A_i \xrightarrow{u_i} A \xrightarrow{v_i} A_i$$

such that

$$v_i u_i = 1_{A_i}, \quad v_i u_j = 0 \quad \text{for } i \neq j.$$

Then the following properties are equivalent:

(i) The u_i yield a direct sum;

(ii) The v_i yield a direct product;

(iii) $\sum_i u_i v_i = 1_A$.

Proof. (i) $\Longrightarrow$ (iii). Let $w = \Sigma u_i v_i$. Then

$$w u_i = (\Sigma u_j v_j) u_i = u_i 1_{A_i} = u_i = 1_A u_i.$$

Thus $w = 1_A$.

(iii) $\Longrightarrow$ (i). Consider

$$H(A,X) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \prod H(A_i,X)$$

given by $\varphi w = \{w u_i\}$ and $\psi\{w_i\} = \sum_i w_i v_i$. Then $\psi\varphi w = \psi\{w u_i\} = \Sigma w u_i v_i = w \Sigma u_i v_i$ $= w 1_A = w$, and $\varphi\psi\{w_i\} = \varphi(\sum_j w_j v_j) = \{(\Sigma w_j v_j) u_i\} = \{w_i v_i u_i\} = \{w_i\}$. Thus φ is an isomorphism, and (i) holds.

The equivalence (ii) $\Longleftrightarrow$ (iii) is dual.

Axiom (A.3) asserts the existence of a direct sum (and direct product) for any two factors. This implies the existence for any finite number of factors. There are easy examples where infinite direct sums or products do not exist. Associativity laws could be proved whenever sums (or products) exist.

Proposition 2. Let

$$u_i\colon A_i \to A, \quad v_i\colon B_i \to B, \quad z_i\colon C_i \to C, \quad i \in I$$

be direct sum representations. Suppose that for each $i \in I$ we have maps

$$A_i \xrightarrow{a_i} B_i \xrightarrow{b_i} C_i.$$

The sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

is exact (where a and b are defined by a_i, b_i) if and only if each of the sequences

$$A_i \xrightarrow{a_i} B_i \xrightarrow{b_i} C_i \longrightarrow 0$$

is exact.

Proof. The sequence $A \to B \to C \to 0$ is exact if and only if the sequence

$$0 \to H(C,X) \to H(B,X) \to H(A,X)$$

is exact for every X. This is equivalent with the exactness of the sequence

$$0 \to \prod H(C_i,X) \to \prod H(B_i,X) \to \prod H(A_i,X)$$

which in turn is equivalent with the exactness of each of the sequences

$$0 \to H(C_i,X) \to H(B_i,X) \to H(A_i,X).$$

The last condition is equivalent with the exactness of

$$A_i \to B_i \to C_i \to 0.$$

Pas très vrai avec des zeros à droite

Proposition 3. Let

$$A' \underset{v'}{\overset{u'}{\rightleftarrows}} A \xrightarrow{v''} A''$$

be such that $v'u' = 1_{A'}$. Then the following conditions are equivalent:

(i) The sequence $0 \to A' \xrightarrow{u'} A \xrightarrow{v''} A'' \to 0$ is exact.

(ii) There exists a map $u''\colon A'' \to A$ such that $v''u'' = 1_{A''}$, $u'v' + u''v'' = 1_A$, $v''u' = 0$, $v'u'' = 0$.

(iii) The maps v', v'' yield a direct product representation of A.

(iv) For every X the sequence

$$0 \to H(X,A') \to H(X,A) \to H(X,A'') \to 0$$

induced by u' and v'' is exact.

Proof. (i) $\Longrightarrow$ (ii). It follows from (i) that the sequence

$$0 \to H(A'',A) \to H(A,A) \to H(A',A)$$

is exact. In $H(A,A)$ consider the element $1_A - u'v'$. Its image in $H(A',A)$ is

$$(1_A - u'v')u' = u' - u'v'u' = u' - u' = 0.$$

Thus there is an element $u'' \in H(A'',A)$ such that $u''v'' = 1_A - u'v'$. Consequently

$$(*) \qquad u'v' + u''v'' = 1_A.$$

Applying v'' on the left we find

$$v''u''v'' = v''.$$

Since v'' is an epimorphism we have

$$u''v'' = 1_{A''}.$$

Multiplying (A) on the right by u'' we obtain $u'v''u'' + u''v''u'' = u''$. Since $u''v'' = 1_{A''}$, it follows that $u'v''u'' = 0$. Since u' is a monomorphism, we have $v''u'' = 0$. Thus (ii) holds.

(ii) $\Longrightarrow$ (iii) follows from proposition 1.

(iii) $\Longleftrightarrow$ (iv). Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & H(X,A') & \to & H(X,A) & \to & H(X,A'') & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ 0 & \to & H(X,A') & \to & H(X,A')+H(X,A'') & \to & H(X,A'') & \to & 0 \end{array}$$

where $\varphi z = (v'z, v''z)$ and the remaining maps are the obvious ones. The second square is always commutative. The first square is commutative because $v'u' = 1_{A'}$. Since the lower row is exact, it follows that the upper row is exact if and only if φ is an isomorphism, i.e. if and only if (iii) holds.

(iv) $\Longrightarrow$ (i). Since (iv) holds for each X it follows that the sequence $0 \to A' \xrightarrow{u'} A \xrightarrow{v''} A''$ is exact. Thus we only need to show that v'' is an epimorphism. This follows from (iii) and elementary property 2.

Proposition 4. Given

$$u'\colon A' \to A$$

the following properties are equivalent:

(i) There exists $v'\colon A \to A'$ such that $v'u' = 1_{A'}$.

(ii) There exists $u''\colon A' \to A$ such that u', u'' yield a direct sum decomposition of A.

(iii) $H(A,X) \to H(A',X)$ induced by u' is an epimorphism for all X.

Proof. (i) $\Longrightarrow$ (ii). Given u', v', let $v''\colon A \to A''$ be the cokernel of u'. Then the sequence $A' \xrightarrow{u'} A \xrightarrow{v''} A'' \to 0$ is exact. Since $v'u' = 1_{A'}$, it follows that u' is a monomorphism and thus (i) of Prop. 3 holds. Consequently (ii) holds and u', u'' are a direct sum representation of A.

(ii) $\Longrightarrow$ (iii). This follows from the dual of Prop. 3, implication (iii) $\Longrightarrow$ (iv).

(iii) $\Longrightarrow$ (i). Since $H(A,A') \to H(A',A')$ is an epimorphism there exists $v' \in H(A,A')$ such that $v'u' = 1_{A'}$.

Definition. If the conditions (i) - (iii) of Prop. 4 are satisfied, we shall say that $u': A' \to A$ is a direct summand of A.

5. Split exact sequences. scindée, fissible

Consider an exact sequence A

(1) $$\dots \to A_i \xrightarrow{u_i} A_{i-1} \xrightarrow{u_{i-1}} \dots, \qquad -\infty < i < \infty.$$

For each i, consider a factorization

$$A_i \xrightarrow{v_i} Z_{i-1} \xrightarrow{w_{i-1}} A_i, \qquad u_i = w_{i-1}v_i,$$

where v_i is an epimorphism and w_{i-1} is a monomorphism. There result exact sequences

(2) $$0 \to Z_i \xrightarrow{w_i} A_i \xrightarrow{v_i} Z_{i-1} \to 0.$$

Proposition 1. The following conditions are equivalent:

(i) For each $X \in \mathcal{A}$ the sequence $H(X,A)$ is exact.

(i*) For each $X \in \mathcal{A}$ the sequence $H(A,X)$ is exact.

(ii) Each v_i is a direct factor of A_i.

(ii*) Each w_i is a direct factor of A_i.

(iii) There exist maps $s_i: A_i \to A_{i+1}$ $(-\infty < i < \infty)$ such that

$$u_{i+1}s_i + s_{i-1}u_i = 1_{A_i}.$$

Proof. (i) $\Longrightarrow$ (ii). We consider the exact sequence

$$H(Z_i,A_{i+1}) \to H(Z_i,A_i) \to H(Z_i,A_{i-1})$$

and the element $w_i \in H(Z_i,A_i)$. We have $u_iw_i = 0$ since $w_i = \mathrm{Ker}\, u_i$. Thus w_i is in the image of $H(Z_i,A_{i+1})$, and there exists $z \in H(Z_i,A_{i+1})$ such that $u_{i+1}z = w_i$. Thus

$$w_i = u_{i+1}z = w_iv_{i+1}z$$

and since w_i is a monomorphism

$$v_{i+1}z = 1_{Z_i}.$$

Thus, by 4, prop. 4*, v_{i+1} is a direct factor.

(ii) $\Longrightarrow$ (iii). In view of 4, Prop. 3* there exist maps

$$Z_i \xleftarrow{\overline{w}_i} A_i \xleftarrow{\overline{v}_i} Z_{i-1}$$

such that

$$\bar{w}_i w_i = 1_{Z_i}, \quad v_i \bar{v}_i = 1_{Z_{i-1}}, \quad w_i \bar{w}_i + \bar{v}_i v_i = 1_{A_i}.$$

Set

$$s_i = \bar{v}_{i+1} \bar{w}_i : A_i \longrightarrow A_{i+1}.$$

We have

$$u_{i+1} s_i + s_{i-1} u_i = w_i v_{i+1} \bar{v}_{i+1} \bar{w}_i + \bar{v}_i \bar{w}_{i-1} w_{i-1} v_i = w_i \bar{w}_i + \bar{v}_i v_i = 1_{A_i}$$

as required.

(iii) $\Longrightarrow$ (i). For $X \in \mathcal{A}$ consider the maps

$$H(X, A_{i+1}) \xrightarrow{u_{i+1}^!} H(X, A_i) \xrightarrow{u_i^!} H(X, A_{i-1})$$

with $u_{i+1}^!$, $u_i^!$ induced by u_{i+1}, u_i. Given $a : X \to A_{i+1}$ we have

$$u_i^! \, u_{i+1}^! (a) = u_{i+1} u_i a = 0$$

so that $u_i^! \, u_{i+1}^! = 0$. Now let $b \in H(X, A_i)$ and $u_i^! b = 0$. Then

$$b = (u_{i+1} s_i + s_{i-1} u_i) b = u_{i+1}^! (sb).$$

Thus $\operatorname{Im} u_{i+1}^! = \operatorname{Ker} u_i^!$ and the sequence $H(X, A)$ is exact.

Having proved the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) the remainder of the proposition follows since (iii) is a self-dual statement.

Definition. If the conditions of Prop. 1, we say that the exact sequence **A** splits. The maps $\{s_i\}$ such as in (iii) are called splitting maps for **A**. *applicatia de scissia*

Corollary. The sequence **A** splits if and only if each of the sequences (2) splits.

Proposition 2. Let

$$0 \to A' \xrightarrow{u'} A \xrightarrow{u''} A'' \to 0, \quad 0 \to B' \xrightarrow{v'} B \xrightarrow{v''} B'' \to 0$$

be exact sequences that split and let

$$A' \xleftarrow{s'} A \xleftarrow{s''} A'' \qquad B' \xleftarrow{t'} B \xleftarrow{t''} B''$$

be appropriate splitting maps. A diagram

$$\begin{array}{ccccccccc} 0 & \to & A' & \xrightarrow{u'} & A & \xrightarrow{u''} & A'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \to & B' & \xrightarrow[v']{} & B & \xrightarrow[v'']{} & B'' & \longrightarrow & 0 \end{array}$$

Se donner f' et f'' alors on a la ~~formule~~ formule donnant f is commutative if and only if

$$(*) \qquad f = v'f's' + t''f''u'' + v'gu''$$

for some $g\colon A'' \to B'$. If this is the case then

$$g = t'fs''.$$

Proof. Assume the diagram is commutative. Then

$$\begin{aligned} f &= f(u's' + s''u'') = v'f's' + fs''u'' \\ &= v'f's' + (v't' + t''v'')fs''u'' \\ &= v'f's' + t''f''u''s''u'' + v't'fs''u''. \end{aligned}$$

Since

$$u'' = u''(u's' + s''u'') = u''s''u''$$

we find that f satisfies $(*)$ with $g = t'fs''$.

Conversely assume that $(*)$ holds. Then

$$\begin{aligned} fu' &= v'f's'u' + t''f''u''u' + v'gu''u' \\ &= v'f's'u' = v'f' \\ v''f &= v''v'f's' + v''t''f''u'' + v''v'gu'' \\ &= v''t''f''u'' = f''u'' \end{aligned}$$

since $s'u' = 1_{A'}$, $v''t'' = 1_{B''}$. Thus the diagram is commutative. Moreover

$$t'fs'' = t'v'f's's'' + t't''f''u''s'' + t'v'gu''s''.$$

Now

$$s's'' = s'(u's' + s''u'')s'' = s's'' + s's''$$

because $s'u'$ and $u''s''$ are identity maps. Thus $s's'' = 0$. Similarly $t't'' = 0$. Thus

$$t'fs'' = t'v'gu''s'' = g$$

because $t'v'$ and $u''s''$ are identities. This concludes the proof.

6. Subgadgets.

For each object A in a category $\mathcal{A}$ we have defined the ordered classes M(A) and E(A). These were the monomorphism classes $u\colon B \to A$ resp. the epimorphism classes $u\colon A \to B$. We shall now see that in the case of an abelian category $\mathcal{A}$, the classes M(A) and E(A) essentially coincide.

Proposition. Consider exact sequences

$$0 \to A' \xrightarrow{u} A \xrightarrow{v} A'' \to 0$$

$$0 \to A_1' \xrightarrow{u_1} A \xrightarrow{v_1} A_1'' \to 0.$$

Then the following conditions are equivalent:

(1) $u < u_1$ (in $M(A)$)

(2) $v_1 > v$ (in $E(A)$)

(3) $v_1 u = 0.$

Proof. (1) $\Longrightarrow$ (3). $u_1 < u$ means that there exists $w: A' \to A_1'$ such that $u = u_1 w$. Then $v_1 u = v_1 u_1 w = 0$.

(3) $\Longrightarrow$ (1). Consider the exact sequence

$$0 \to H(A', A_1') \xrightarrow{\psi} H(A', A) \xrightarrow{\varphi} H(A', A_1'').$$

For $u \in H(A', A)$ we have $\varphi u = v_1 u = 0$. Thus there exists an element $w \in H(A', A_1')$ such that $\psi w = u$. Thus $u_1 w = u$ and $u < u_1$.

The proof of (2) $\Longleftrightarrow$ (3) is dual.

In view of this proposition we find that the relation $u \longleftrightarrow v$ establishes a 1-1 order reversing correspondence between $M(A)$ and $E(A)$. If we reverse the order in $E(A)$ we can identify this class with $M(A)$. We shall call $M(A)$ the <u>class of subobjects</u> of A. Any subobject of A may thus be represented by a monomorphism $u: B \to A$ or an epimorphism $v: A \to C$. When we pass to the dual category, inequalities are reversed.

Proposition. The class $M(A)$ is a lattice.

Proof. Given $u_1, u_2 \in M(A)$ represented by monomorphisms

$$u_1: B_1 \to A \qquad u_2: B_2 \to A$$

consider the direct sum of B_1 and B_2 represented by

$$v_1: B_1 \to B, \qquad v_2: B_2 \to B.$$

There exists then a unique map

$$u: B \to A$$

such that $uv_1 = u_1$, $uv_2 = u_2$. Let

$$B \xrightarrow{u'} C \xrightarrow{u''} A, \qquad u = u''u'$$

where u' is an epimorphism and u'' is a monomorphism. Then $u''u'v_1 = u_1$ and $u''u'v_2 = u_2$. Thus $u_1 < u''$, $u_2 < u''$.

Suppose that $u_1 < w$, $u_2 < w$. Represent w as an epimorphism $w: A \to D$. Then by Prop. $wu_i = 0$, $i = 1,2$. This implies $wuv_i = 0$, $i = 1,2$ and therefore $wu = 0$. Thus $wu''u' = 0$. Since u' is an epimorphism it follows that $wu'' = 0$. Thus $u'' < w$. This proves that $u'' = u_1 \cup u_2$.

The existence of $u_1 \cap u_2$ is dual.

It is clear that the existence of $\bigcup_i u_i$ for $i \in I$ depends only on the existence of the direct sum of A_i, $i \in I$.

Given

$$f: A \to B$$

in $\mathcal{A}$ we shall define maps

$$f^+: M(A) \to M(B), \qquad f^-: M(B) \to M(A)$$

as follows. Given a monomorphism $u: A' \to A$ define $f^+(u) = \text{Im } fu$. Given an epimorphism $v: A \to A''$ define $f^-(v) = \text{Coker } vf$.

The sorite concerning f^+, f^-, $u_1 \cup u_2$, $u_1 \cap u_2$ is omitted.

Ajouter : th. d'isomorphismes de Mlle Noether.

I devrait être : <u>catégories et foncteurs</u>

Rien que des "~~applic.~~ morphismes" c. à d. système algébrique | composition pas partout définie, associativité, identités (identifiées aux "objets")

D'où des $H(A,B)$ ("morphismes de A dans B)

catégorie abélienne : $H(A,B)$ est un <u>groupe abélien</u>

catégorie <u>duale</u> : | renverser les flèches
$H(A,B) \to H(B,A)$

monomorphisme / épimorphisme | sont définis dans les catégories générales

$X \rightrightarrows^{u}_{v} A \xrightarrow{m} B$ (mono) $m\circ u = m\circ v \Rightarrow u = v$ (qqsoient X, u, v)

$A \xrightarrow{e} B \rightrightarrows^{u}_{v} Y$ $u\circ e = v\circ e \Rightarrow u = v$ (epi)

$$\begin{array}{ccc} B & & \\ {\scriptstyle u}\downarrow & \searrow^{\varphi} & \\ & & A \\ B' & \nearrow_{\varphi'} & \end{array}$$

φ, φ' : monomorphismes
Si u rend le diagramme commutatif, alors u est un isomorphisme, est unique
$\Longrightarrow$ classe de monomorphismes.

<u>Lemme abstrait des cinq.</u> On prend un X t.q. $X \to C \to C'$ soit 0

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & & \downarrow & & & & \\ A & \to & B & \to & C & \to & D & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

III devrait être l'<u>homologie abstraite</u> (complexes, ...)

chap. IV et XV du [illegible] traduits en catégories additives.

V sera très joli

IV. RESOLUTIONS

1. Classes of exact sequences.

Let $\mathcal{A}$ be an abelian category and let $\mathcal{E}$ be a class of exact sequences (ranging from $-\infty$ to $+\infty$) in $\mathcal{A}$. An object $A \in \mathcal{A}$ will be called $\mathcal{E}$-projective provided the sequence $H(A,E)$ is exact for every $E \in \mathcal{E}$. The class of all $\mathcal{E}$-projective objects will be denoted by $\mathcal{E}-\mathcal{P}$.

We now define the class $\bar{\mathcal{E}}$ of exact sequences as follows: $E \in \bar{\mathcal{E}}$ provided $H(A,E)$ is exact for every $A \in \mathcal{E}-\mathcal{P}$. Clearly $\mathcal{E} \subset \bar{\mathcal{E}}$ and $\mathcal{E}-\mathcal{P} = \bar{\mathcal{E}}-\mathcal{P}$. Therefore $\bar{\bar{\mathcal{E}}} = \bar{\mathcal{E}}$. If $\mathcal{E} = \bar{\mathcal{E}}$ then we say that the class $\mathcal{E}$ is closed. Henceforth we shall assume that $\mathcal{E}$ is a closed class of exact sequences in $\mathcal{A}$.

Proposition 1. Consider exact sequences

$$(E_n) \qquad 0 \to Z_n \to A_n \to Z_{n-1} \to 0$$

for $-\infty < n < \infty$. Let

$$(E) \qquad \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$$

be the sequence defined by compositions

$$A_{n+1} \to Z_n \to A_n.$$

Then the sequence E is in $\mathcal{E}$ if and only if each E_n is in $\mathcal{E}$.

Proof. Let A be $\mathcal{E}$-projective. Each sequence

$$0 \to H(A,Z_n) \to H(A,A_n) \xrightarrow{\varphi_n} H(A,Z_{n-1})$$

is exact. It is now clear that the sequence $H(A,E)$ is exact if and only if each φ_n is an epimorphism. Thus $E \in \mathcal{E}$ if and only if $E_n \in \mathcal{E}$ for each n.

Proposition 2. Let E and E_i $(i \in I)$ be exact sequences and let

$$v_i: E \to E_i, \quad i \in I$$

be a family of maps which yield direct product representations for each index n. Then $E \in \mathcal{E}$ if and only if each $E_i \in \mathcal{E}$.

Proof. Let $A \in \mathcal{E}-\mathcal{P}$. Then we have the isomorphism

$$H(A,E) \approx \prod_{i \in I} H(A,E_i).$$

Thus the sequence $H(A,E)$ is exact if and only if each of the sequences $H(A,E_i)$ is exact.

Proposition 3. Let

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of exact sequences, and assume that for each n the sequence

$$0 \to E_n' \to E_n \to E_n'' \to 0$$

is in $\mathcal{E}$. If two of the sequences E, E', E'' are in $\mathcal{E}$, then so is the third.

Proof. Let A be $\mathcal{E}$-projective. Then each of the sequences

$$0 \to H(A,E_n') \to H(A,E_n) \to H(A,E_n'') \to 0$$

is exact. It follows that the sequence of complexes

$$0 \to H(A,E') \to H(A,E) \to H(A,E'') \to 0$$

is exact. Thus we obtain an exact diagram

$$\begin{array}{ccc} \mathcal{H}(H(A,E)) & \to & \mathcal{H}(H(A,E'')) \\ & \nwarrow \quad \mathcal{H}(H(A,E')) \quad \swarrow \partial & \end{array}$$

of homology groups. Thus if two of the homology groups are zero, so is the third.

Proposition 4. Let $v_i: A_i \to A$, $i \in I$ be a direct sum representation of A. Then A is $\mathcal{E}$-projective if and only if each A_i is $\mathcal{E}$-projective.

Proof. Let E be any exact sequence in $\mathcal{E}$. Then the map

$$H(A,E) \to \prod H(A_i,E)$$

is an isomorphism. Thus $H(A,E)$ is exact if and only if each $H(A_i,E)$ is exact.

Proposition 5. If $0 \to A' \xrightarrow{u'} A \xrightarrow{v''} A'' \to 0$ is an exact sequence in $\mathcal{E}$ and A'' is $\mathcal{E}$-projective then the exact sequence splits (i.e. u' is a direct summand and (or) v'' is a direct factor).

Proof. The sequence

$$0 \to H(A'',A') \to H(A'',A) \to H(A'',A'') \to 0$$

is exact. Therefore there exists a map $u'': A'' \to A$ such that $v''u'' = 1_{A''}$. Thus v'' is a direct factor.

2. Resolutions.

Consider a sequence

$$(1) \qquad \cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_o \xrightarrow{\varepsilon} A \to 0$$

in $\mathcal{A}$ with

$$\varepsilon d_1 = 0, \quad d_i d_{i+1} = 0 \quad \text{for } i = 1,2,\ldots$$

The sequence (1) may also be written in the form of a commutative diagram

$$(2) \qquad \begin{array}{ccccccccccc} \cdots \to & X_n & \to & X_{n-1} & \to \cdots \to & X_1 & \xrightarrow{d_1} & X_o & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \varepsilon & & \\ \cdots \to & 0 & \to & 0 & \to \cdots \to & 0 & \to & A & \to & 0. \end{array}$$

The top row yields a complex which will be denoted by X, the bottom row is a complex which may be identified with A. The ε: X $\to$ A is a map of complexes. We shall say that X is a complex over A with ε as augmentation.

We shall say that X is $\mathcal{E}$-acyclic if (1) is an exact sequence in $\mathcal{E}$. We shall say that X is $\mathcal{E}$-projective if each X_i $(i = 1,2,\ldots)$ is $\mathcal{E}$-projective. If X (together with ε) is both $\mathcal{E}$-acyclic and $\mathcal{E}$-projective then we say that X is an $\mathcal{E}$-projective resolution of A (with ε as augmentation).

Proposition 1. Consider a diagram

$$\begin{array}{ccc} X & & X' \\ \varepsilon \downarrow & & \downarrow \varepsilon' \\ A & \xrightarrow[f]{} & A' \end{array}$$

where X (resp. X') is a complex over A (resp. A') with augmentation ε (resp. ε'). If X is $\mathcal{E}$-projective and X' is $\mathcal{E}$-acyclic, then there exists a map F: X $\to$ X' such that $\varepsilon' F = f\varepsilon$. Any two such maps F are homotopic.

A map F as above will be called a map <u>over</u> f.

Proof. Since X_o is $\mathcal{E}$-projective and X' is $\mathcal{E}$-acyclic, it follows that the sequence

$$H(X_o, X_o') \to H(X_o, A') \to 0$$

is exact. Thus there exists an element $F_o \in H(X_o, X_o')$ which maps onto $f\varepsilon \in H(X_o, A')$. Thus $\varepsilon' F_o = f\varepsilon$. Assume now that F_i: $X_i \to X_i'$ are defined for $i = 0,\ldots,n-1$ $(n > 1)$ and that

$$d_i'F_i = F_{i-1}d_i \quad \text{for } i = 1,\dots,n-1$$

$$\varepsilon'F_o = f\varepsilon.$$

Since X_n is $\mathcal{E}$-projective and X' is $\mathcal{E}$-acyclic we have the exact sequence

$$H(X_n,X_n') \to H(X_n,X_{n-1}') \to H(X_n,X_{n-2}')$$

(if $n = 1$, replace X_{n-2}' by A'). Consider the element $F_{n-1}d_n \in H(X_n,X_{n-1}')$. Its image in $H(X_n,X_{n-2}')$ is

$$d_{n-1}'F_{n-1}d_n = F_{n-2}d_{n-1}d_n = 0 \quad \text{if } n > 1$$

$$\varepsilon'F_od_1 = f\varepsilon d_1 = 0 \quad \text{if } n = 1.$$

Thus there exists F_i: $X_i \to X_i'$ with $d_i'F_i = F_{i-1}d_i$. This proves the existence of F.

Suppose now that F,G: $X \to X'$ are two maps over f: $A \to A'$. We shall define a homotopy S: $F \simeq G$. Clearly S_i: $X_i \to X_{i+1}'$ is zero for $i < 0$. As above we have the exact sequence

$$H(X_o,X_1') \to H(X_o,X_o') \to H(X_o,A').$$

Consider the element $G_o - F_o \in H(X_o,X_o')$. Its image in $H(X_o,A')$ is

$$\varepsilon'(G_o - F_o) = f\varepsilon - f\varepsilon = 0.$$

Thus there is $S_o \in H(X_o,X_1')$ with $d_1'S_o = G_o - F_o$. Consequently

$$d_1'S_o + S_{-1}d_o = G_o - F_o$$

since $S_{-1} = 0$. Assume now that S_i: $X_i \to X_{i+1}'$ is already defined for $i < n$ ($n > 0$) and that

(*) $$d_{i+1}'S_i + S_{i-1}d_i = G_i - F_i \quad \text{for } i < n.$$

Consider the exact sequence

$$H(X_n,X_{n+1}') \to H(X_n,X_n') \to H(X_n,X_{n-1}')$$

and consider the element

$$u = G_n - F_n - S_{n-1}d_n \in H(X_n,X_n').$$

We have

$$d_n'u = d_n'G_n - d_n'F_n - d_n'S_{n-1}d_n =$$

$$= G_nd_n - F_nd_n - (G_n - F_n - S_{n-2}d_{n-1})d_n = 0.$$

Thus there exists $S_n \in H(X_n, X'_{n+1})$ such that $d'_{n+1}S_n = u$. It follows that (A) holds for $i = n$. This completes the proof.

3. Resolutions of exact sequences.

Proposition 1. Consider the diagram

$$\begin{array}{ccccccccc} & & X' & & & & X'' & & \\ & & \downarrow \varepsilon' & & & & \downarrow \varepsilon'' & & \\ 0 & \to & A' & \xrightarrow[f]{} & A & \xrightarrow[g]{} & A'' & \to & 0 \end{array}$$

where the row is $\mathcal{E}$-exact, X' is an $\mathcal{E}$-acyclic complex over A', X'' is an $\mathcal{E}$-projective complex over A''. There exists a complex X over A and maps such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \xrightarrow{F} & X & \xrightarrow{G} & X'' & \to & 0 \\ & & \varepsilon' \downarrow & & \varepsilon \downarrow & & \varepsilon'' \downarrow & & \\ 0 & \to & A' & \xrightarrow[f]{} & A & \xrightarrow[g]{} & A'' & \to & 0 \end{array}$$

is commutative and such that each row

$$0 \to X_n' \xrightarrow{F_n} X_n \xrightarrow{G_n} X_n'' \to 0 \qquad \text{(A)}$$

is exact and splits.

Proof. For each n, let X_n be a direct sum of X_n' and X_n''. There results a split exact sequence

$$0 \to X_n' \xrightarrow{F_n} X_n \xrightarrow{G_n} X_n'' \to 0$$

with splitting maps

$$X_n' \xleftarrow{\bar{F}_n} X_n \xleftarrow{\bar{G}_n} X_n''.$$

Let $\Sigma_n : X_n'' \to X'_{n-1}$, $n = 1,2,\ldots$ be any sequence of maps. Setting

$$d_n = F_{n-1}d_n'\bar{F}_n + \bar{G}_{n-1}d_n''G_n + F_{n-1}\Sigma_n G_n$$

we obtain maps $d_n : X_n \to X_{n-1}$ and by II,4, prop. 2 the diagrams

$$\begin{array}{ccccccccc} 0 & \to & X_n' & \to & X_n & \to & X_n'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X'_{n-1} & \to & X_n & \to & X''_{n-1} & \to & 0 \end{array}$$

are commutative.

Further let $\sigma : X_0'' \to A$ be any map, and define

$$\varepsilon = f\varepsilon'\bar{F}_0 + \sigma G_0 : X \to A.$$

Then

$$\varepsilon F_o = f\varepsilon' \bar{F}_o F_o + \sigma G_o F_o = f\varepsilon'$$

since $\bar{F}_o F_o = 1_{X_o'}$, $G_o F_o = 0$. Thus the diagram

$$\begin{array}{ccc} X_o' & \xrightarrow{F_o} & X_o \\ \varepsilon' \downarrow & & \downarrow \varepsilon \\ A' & \xrightarrow[f]{} & A \end{array}$$

is commutative. Further

$$\varepsilon \bar{G}_o = f\varepsilon' \bar{F}_o \bar{G}_o + \sigma G_o \bar{G}_o = \sigma$$

because $\bar{F}_o \bar{G}_o = 0$ and $G_o \bar{G}_o = 1_{X_o''}$. Conversely if $\varepsilon\colon X_o \to A$ is such that $\varepsilon F_o = f\varepsilon'$ then

$$\varepsilon = \varepsilon(F_o \bar{F}_o + \bar{G}_o G_o) = f\varepsilon' \bar{F}_o + \varepsilon \bar{G}_o G_o = f\varepsilon' \bar{F}_o + \sigma G_o$$

if we set $\sigma = \varepsilon \bar{G}_o$.

The commutativity in the diagram

$$\begin{array}{ccc} X_o & \longrightarrow & X_o'' \\ \varepsilon \downarrow & & \downarrow \varepsilon'' \\ A & \longrightarrow & A'' \end{array}$$

and the conditions $\varepsilon d_1 = 0$, $d_{n-1} d_n = 0$ for $n > 0$, are then equivalent with the conditions

$$\text{(i)} \qquad \left\{ \begin{aligned} g\sigma &= \varepsilon'' \\ f\varepsilon' \Sigma_1 + \sigma d_1'' &= 0 \\ d_{n-1}' \Sigma_n + \Sigma_{n-1} d_n'' &= 0. \end{aligned} \right.$$

The problem thus reduces to finding σ and $\{\Sigma_n\}$ satisfying equations (i). These will be solved step by step as follows. First condider the sequence

$$H(X_o'', A) \to H(X_o'', A'') \to 0$$

which is exact because X_o'' is $\mathcal{E}$-projective and $0 \to A' \to A \to A'' \to 0$ is in $\mathcal{E}$. There is then $\sigma \in H(X_o'', A)$ such that $g\sigma = \varepsilon''$.

From the exact sequences

$$0 \to H(X_1'', A') \to H(X_1'', A) \to H(X_1'', A'') \to 0$$
$$\cdots \to H(X_1'', X_o') \to H(X_1'', A') \to 0$$

we deduce the exact sequence

$$H(X_1'',X_0') \to H(X_1'',A) \to H(X_1'',A'') \to 0.$$

Consider the element $u = -\sigma d_1'' \in H(X_1'',A)$. Its image in $H(X_1'',A'')$ is $gu = -g\sigma d_1'' = -\varepsilon'' d_1'' = 0$. Thus there exists $\Sigma_1 \in H(X_1'',X_0')$ with $f\varepsilon'\Sigma_1 = u = -\sigma d_1''$. Next consider the exact sequence

$$H(X_2'',X_1') \to H(X_2'',X_0') \to H(X_2,A').$$

In $H(X_2'',X_0')$ consider the element $u = -\Sigma_1 d_2''$. Its image in $H(X_2,A')$ is $\varepsilon' u = -\varepsilon'\Sigma_1 d_2''$. We have

$$f\varepsilon' u = -f\varepsilon'\Sigma_1 d_2'' = \sigma d_1'' d_2'' = 0.$$

Since f is a monomorphism it follows that $\varepsilon' u = 0$ and thus there exists $\Sigma_2 \in H(X_2'',X_1')$ with $d_1'\Sigma_2 = u$. Thus $d_1'\Sigma_2 + \Sigma_1 d_2'' = 0$ as desired.

Finally assume by induction that $\sigma, \Sigma_1, \Sigma_2, \ldots, \Sigma_{n-1}$ are defined for some $n > 2$ and that equations (i) hold. Consider the exact sequence

$$H(X_n'',X_{n-1}') \to H(X_n'',X_{n-2}') \to H(X_n'',X_{n-3}')$$

and in $H(X_n'',X_{n-2}')$ take the element $u = -\Sigma_{n-1}d_n''$. Its image in $H(X_n'',X_{n-3}')$ is $d_{n-2}'u = -d_{n-2}'\Sigma_{n-1}d_n'' = \Sigma_{n-2}d_{n-1}''d_n'' = 0$. Thus there exists $\Sigma_n \in H(X_n'',X_{n-1}')$ such that $d_{n-1}'\Sigma_n = u$. Then $d_{n-1}'\Sigma_n + \Sigma_{n-1}d_n'' = 0$, and the proof is complete.

Corollary. If X' and X'' are $\mathcal{E}$-projective resolutions of A' and A'' then X is an $\mathcal{E}$-projective resolution of A.

Indeed, since each X_i is a direct sum of X_i' and X_i'' it follows from 1, prop. 4 that X_i is $\mathcal{E}$-projective. There remains to verify that the sequence

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_o \to A \to 0$$

is in $\mathcal{E}$. This follows from 1, prop. 3.

Definition. Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \to & X & \to & X'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & A & \to & A'' & \to & 0 \end{array}$$

with exact rows. If X (resp X', resp X'') is an $\mathcal{E}$-projective resolution of A (resp A', resp A'') then $0 \to X' \to X \to X'' \to 0$ is said to be an $\mathcal{E}$-projective resolution of $0 \to A' \to A \to A'' \to 0$.

Proposition 2. Let

$$\begin{array}{ccccccccc} 0 & \to & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \to & 0 \\ & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' & & \\ 0 & \to & B' & \xrightarrow[f^*]{} & B & \xrightarrow[g^*]{} & B'' & \to & 0 \end{array}$$

be a commutative diagram with exact rows. Let

$$0 \to X' \xrightarrow{F} X \xrightarrow{G} X'' \to 0, \qquad 0 \to Y' \xrightarrow{F^*} Y \xrightarrow{G^*} Y'' \to 0$$

be $\mathcal{E}$-projective resolutions of the exact sequences

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0, \qquad 0 \to B' \xrightarrow{f^*} B \xrightarrow{g^*} B'' \to 0.$$

Then there exist maps

$$\Phi' : X' \to Y', \qquad \Phi : X \to Y, \qquad \Phi'' : X'' \to Y''$$

over φ', φ, φ'' respectively such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \to & X & \to & X'' & \to & 0 \\ & & \downarrow \Phi' & & \downarrow \Phi & & \downarrow \Phi'' & & \\ 0 & \to & Y' & \to & Y & \to & Y'' & \to & 0 \end{array}$$

is commutative.

If $\psi' : X' \to Y'$, $\psi : X \to Y$, $\psi'' : X'' \to Y''$ is another triple of maps with the same property then there exists a triple of homotopies:

$$S' : \Phi' \simeq \psi', \qquad S : \Phi \simeq \psi, \qquad S'' : \Phi'' \simeq \psi''$$

such that for each n the diagram

$$\begin{array}{ccccccccc} 0 & \to & X_n' & \to & X_n & \to & X_n'' & \to & 0 \\ & & \downarrow S_n' & & \downarrow S_n & & \downarrow S_n'' & & \\ 0 & \to & Y_{n+1}' & \to & Y_{n+1} & \to & Y_{n+1}'' & \to & 0 \end{array}$$

is commutative.

Proof. Let

$$X_n' \underset{\bar{F}_n}{\overset{F_n}{\rightleftarrows}} X_n \underset{\bar{G}_n}{\overset{G_n}{\rightleftarrows}} X_n''$$

be diagrams with properties as listed at the beginning of the proof of prop. 1. As in that proof we assume that the maps $d_n : X_n \to X_{n-1}$ and $\varepsilon : X_o \to A$ are given by maps

$$\sigma : X_o'' \to A, \qquad \Sigma_n : X_n'' \to X_{n-1}'$$

satisfying condition (i). We shall make similar constructions for the exact sequence $0 \to Y' \to Y \to Y'' \to 0$ with all the homomorphisms carrying an asterisk.

We choose maps

$$\Phi' : X' \to Y', \quad \Phi'' : X'' \to Y''$$

over φ' and φ'' using Prop. 1. We shall show that $\Phi : X \to Y$ can be constructed so as to satisfy the conditions of the proposition.

A map $\Phi_n : X_n \to Y_n$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & X_n' & \xrightarrow{F_n} & X_n & \xrightarrow{G_n} & X_n'' & \to & 0 \\ & & \downarrow \Phi_n' & & \downarrow \Phi_n & & \downarrow \Phi_n'' & & \\ 0 & \to & Y_n' & \xrightarrow[F_n^*]{} & Y_n & \xrightarrow[G_n^*]{} & Y_n'' & \to & 0 \end{array}$$

must be of the form (II, 4, prop. 2),

$$\Phi_n = F_n^* \Phi'_n \bar{F}_n + \bar{G}_n^* \Phi_n'' G_n + F_n^* \Gamma_n G_n$$

where

$$\Gamma_n : X_n'' \to Y_n'.$$

The commutativity conditions in the diagrams

$$\begin{array}{ccc} X_o & \xrightarrow{\Phi_o} & Y_o \\ \varepsilon \downarrow & & \downarrow \varepsilon^* \\ A & \xrightarrow[\varphi]{} & B \end{array} \qquad\qquad \begin{array}{ccc} X_n & \xrightarrow{\Phi_n} & Y_n \\ d_n \downarrow & & \downarrow d_n^* \\ X_{n-1} & \to & Y_{n-1} \end{array}$$

translate as follows:

$$\text{(ii)} \quad \begin{cases} f^* \varepsilon'^* \Gamma_o = \varphi\sigma - \sigma^* \Phi_o'' \\ d_n' \Gamma_n - \Gamma_{n-1} d_n'' = \Phi_{n-1}' \Sigma_n - \Sigma_n^* \Phi_n''. \end{cases}$$

These equations are solved step by step as in the previous proof. Boring procedure is omitted.

The second part of the proposition can be proved by the same method. However there exists a more conceptual proof which will be given here.

Let

$$\psi' : X' \to Y', \quad \psi : X \to Y, \quad \psi'' : X'' \to Y''$$

be another triple of maps over φ', φ, φ''. By prop. 1 we have homotopies

$$S' : \Phi' \simeq \psi', \quad S'' : \Phi'' \simeq \psi''.$$

Consider the maps

$$T_n = F_{n+1}^* S_n' \bar{F}_n + \bar{G}_{n+1}^* S_n'' G_n : X_n \to Y_{n+1}.$$

There is then a unique map $\bar{\psi} : X \to Y$ over φ such that $T: \Phi \simeq \psi'$, indeed $\bar{\psi} = \Phi + dT + Td$. Since the diagrams

$$\begin{array}{ccccccccc} 0 & \to & X_n' & \to & X_n & \to & X_n'' & \to & 0 \\ & & \downarrow S' & & \downarrow T & & \downarrow S'' & & \\ 0 & \to & Y'_{n+1} & \to & Y_{n+1} & \to & Y''_{n+1} & \to & 0 \end{array}$$

are commutative, we have a triple of homotopies S', T, S for the triples of maps (Φ', Φ, Φ'') and $(\psi', \bar{\psi}, \psi'')$, as required in proposition 2. Now we compare the triple (ψ', ψ, ψ'') with the triple $(\psi', \bar{\psi}, \psi'')$. If we denote $\Omega = \bar{\psi} - \psi$, this amounts to comparing the triple $(0, \Omega, 0)$ with the triple $(0,0,0)$; here $\Omega: X \to Y$ is a map over the zero map $A \to B$ and the diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \xrightarrow{F} & X & \xrightarrow{G} & X'' & \to & 0 \\ & & 0\downarrow & & \downarrow\Omega & & \downarrow 0 & & \\ 0 & \to & Y' & \xrightarrow[F^*]{} & Y & \xrightarrow[G^*]{} & Y'' & \to & 0 \end{array}$$

is commutative. By the earlier part of the argument we have maps

$$\Gamma_n: X_n'' \to Y_n'$$

such that

$$\Omega_n = F_n^* \Gamma_n G_n.$$

Conditions (i) become

$$\varepsilon'^* \Gamma_o = 0$$

$$d_n' \Gamma_n - \Gamma_{n-1} d_n'' = 0.$$

Thus $\Gamma: X'' \to Y'$ is a map over the zero map $A'' \to B'$ and $\Omega = F^* \Gamma G$. By prop. 1 we have a homotopy $U: \Gamma \simeq 0$. Setting $W: F^* UG$ we obtain a homotopy $W: \Omega \simeq 0$. Since the diagrams

$$\begin{array}{ccccccccc} 0 & \to & X_n' & \xrightarrow{F_n} & X_n & \xrightarrow{G_n} & X_n'' & \to & 0 \\ & & \downarrow 0 & & \downarrow W_n & & \downarrow 0 & & \\ 0 & \to & Y'_{n+1} & \xrightarrow[F^*_{n+1}]{} & Y_{n+1} & \xrightarrow[G^*_{n+1}]{} & Y''_{n+1} & \to & 0 \end{array}$$

are commutative, the proof is complete.

4. Existence of resolutions.

Proposition 1. For every closed family $\mathcal{E}$ of exact sequences in $\mathcal{A}$ the following properties are equivalent:

(1) Every $A \varepsilon \mathcal{A}$ has an $\mathcal{E}$-projective resolution X.

(2) For every $A \varepsilon \mathcal{A}$ there exists an exact sequence

$$0 \longrightarrow Z_o \longrightarrow X_o \longrightarrow A \longrightarrow 0$$

in $\mathcal{E}$ with X_o $\mathcal{E}$-projective.

Proof. (1)$\Longrightarrow$ (2). Let X be an $\mathcal{E}$-projective resolution of A. Then

$$\cdots \longrightarrow X_1 \longrightarrow X_o \longrightarrow A \longrightarrow 0$$

is in $\mathcal{E}$ and X_o is $\mathcal{E}$-projective. (2) now follows from IV, 1, prop. 1.

(2) $\Longrightarrow$ (1). Using (2) construct a sequence

$$0 \rightarrow Z_o \longrightarrow X_o \longrightarrow A_o \longrightarrow 0$$

$$0 \rightarrow Z_1 \longrightarrow X_1 \longrightarrow Z_o \longrightarrow 0$$

$$\cdots$$

$$0 \rightarrow Z_n \longrightarrow X_n \longrightarrow Z_{n-1} \rightarrow 0$$

of sequences in $\mathcal{E}$ with X_n $\mathcal{E}$-projective for $n = 0,1,\ldots$ It follows from IV, 1, prop. 1 that the exact sequence

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \rightarrow X_o \longrightarrow A \longrightarrow 0$$

is in $\mathcal{E}$. Since each X_i is $\mathcal{E}$-projective this yields an $\mathcal{E}$-projective resolution of A.

Definition. A family $\mathcal{E}$ of exact sequences is called <u>complete</u> if it is closed and satisfies the conditions of Prop. 1.

There is a completely dual set of notions obtained by passing to the dual category $\mathcal{A}^*$. Let $\mathcal{E}$ be a family of exact sequences in $\mathcal{A}$ and let $\mathcal{E}^*$ denote the corresponding family in $\mathcal{A}^*$. An element $A \varepsilon \mathcal{A}$ is said to be $\mathcal{E}$-injective if $A \varepsilon \mathcal{A}^*$ is $\mathcal{E}^*$-projective. Similarly for injective resolutions, etc. Thus $A \varepsilon \mathcal{A}$ is $\mathcal{E}$-injective if and only if Hom(B,A) is exact for every exact sequence B in $\mathcal{E}$.

5. Examples.

<u>1</u>. Let $\mathcal{A}$ be an abelian category and let $\mathcal{E}$ be the family of all exact sequences in $\mathcal{A}$ that split. If $A \varepsilon \mathcal{E}$ then it follows from II,5, Prop. 1 that

$H(X,A)$ is exact for every $X \in \mathcal{A}$. Thus every element of $\mathcal{A}$ is $\mathcal{E}$-projective. If A is any exact sequence and if $H(X,A)$ is exact for every $X \in \mathcal{A}$ then again from II,5, Prop. 1 we deduce that A splits. Thus the class $\mathcal{E}$ is closed. Since each $A \in \mathcal{A}$ is $\mathcal{E}$-projective, it follows that $\mathcal{E}$ is complete. For every $A \in \mathcal{A}$ an $\mathcal{E}$-projective resolution of A can be obtained by taking A itself.

2. The most typical case usually considered is when $\mathcal{E}$ is the class of all exact sequences in $\mathcal{A}$. This class obviously is closed. The $\mathcal{E}$-projective elements are frequently called projective (with $\mathcal{E}$ omitted). In general, the class $\mathcal{E}$ is not complete. In Grothendieck's paper two general theorems were given stating sufficient conditions for the completeness of $\mathcal{E}$. In all major applications, fairly simple ad hoc proofs of completeness have been found. Therefore this write-up omits the general theorems, which would require considerable labor and terminology.

3. Let Λ be a ring and let ${}_\Lambda\mathcal{M}$ be the abelian category of left Λ-modules. We take $\mathcal{E}$ to be the class of all exact sequences. In this case the projective modules are exactly the direct summands of free modules. Since each module is isomorphic to the quotient module of a free module it follows that $\mathcal{E}$ is complete.

Now consider the class $\mathcal{E}^*$ of ${}_\Lambda\mathcal{M}^*$. In order to show that this class is complete we must find for each module A a monomorphism $A \to \bar{A}$ where $\bar{A}$ is injective. First consider the case $\Lambda = Z$ (the ring of integers). In this case a module B is injective if and only if $nB = B$ for every integer $n > 0$. Write the module A in the form $A = F/B$, F free. Then define $\bar{A} = (F \otimes R)/B$, where R is the group of all rational numbers. Then $\bar{A}$ is injective and the natural mapping $A \to \bar{A}$ is a monomorphism. Now let Λ be any ring and A a left Λ-module. There exists then a Z-monomorphism $\varphi: A \to \bar{A}$ where $\bar{A}$ is Z-injective. This induces a Λ-monomorphism

$$\mathrm{Hom}_Z(\Lambda, A) \to \mathrm{Hom}_Z(\Lambda, \bar{A}).$$

This combined with the monomorphism

$$A = \mathrm{Hom}_\Lambda(\Lambda, A) \rightarrow \mathrm{Hom}_Z(\Lambda, A)$$

yields a monomorphism

$$A \rightarrow \mathrm{Hom}_Z(\Lambda, \bar{A}).$$

There remains to be proved that the last module is Λ-injective. Let then B be any exact sequence in $_\Lambda\mathfrak{M}$. We have

$$\mathrm{Hom}_\Lambda(B, \mathrm{Hom}_Z(\Lambda, \bar{A})) \approx \mathrm{Hom}_Z(B \otimes_\Lambda \Lambda, \bar{A})$$
$$\approx \mathrm{Hom}_Z(B, \bar{A}).$$

This last sequence is exact because $\bar{A}$ is Z-injective. Thus $\mathrm{Hom}_Z(\Lambda, \bar{A})$ is Λ-injective.

4. Another interesting example can be extracted from a relative homology theory recently proposed by Hochschild. Let

$$\varphi: \Lambda \longrightarrow \Gamma$$

be a ring homomorphism. Let $\mathcal{E}$ be the class of all exact sequences in $_\Gamma\mathfrak{M}$ which split when regarded as exact sequences in $_\Lambda\mathfrak{M}$.

We shall show that both $\mathcal{E}$ and $\mathcal{E}^*$ are closed and complete.

First we show that for any $A \in {_\Lambda\mathfrak{M}}$ the Γ-module $_{(\varphi)}A = \Gamma \otimes_\Lambda A$ is $\mathcal{E}$-projective. Indeed we have for any exact sequence B

$$\mathrm{Hom}_\Gamma(_{(\varphi)}A, B) = \mathrm{Hom}_\Gamma(\Gamma \otimes_\Lambda A, B) \tag{$*$}$$
$$\approx \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_\Gamma(\Gamma, B)) \approx \mathrm{Hom}_\Lambda(A, B).$$

This last sequence is exact if B Λ-splits.

Now we can show that $\mathcal{E}$ is closed. Indeed suppose B is an exact sequence in $_\Gamma\mathfrak{M}$ such that $\mathrm{Hom}_\Gamma(A, B)$ is exact for every A which is $\mathcal{E}$-projective. In particular for every $A \in {_\Lambda\mathfrak{M}}$, the sequence $\mathrm{Hom}_\Gamma(_{(\varphi)}A, B)$ is exact. Thus by ($*$) $\mathrm{Hom}_\Lambda(A, B)$ is exact for every $A \in {_\Lambda\mathfrak{M}}$. This proves (II,5, Prop.1) that B Λ-splits and thus $B \in \mathcal{E}$.

To show that $\mathcal{E}$ is complete take $A \in {_\Gamma\mathfrak{M}}$ and consider the maps

$$A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} {_{(\varphi)}A}$$

given by

$$\alpha a = 1 \otimes_\Lambda a \qquad \beta(\gamma \otimes_\Lambda a) = \gamma a.$$

Clearly $\beta\alpha$ = identity, α is a Γ-monomorphism and β is a Λ-homomorphism. There results an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} {}_{(\varphi)}A \longrightarrow C \longrightarrow 0$$

which Λ-splits. Since ${}_{(\varphi)}A$ is $\mathcal{E}$-projective, it follows that $\mathcal{E}$ is complete.

The treatment of $\mathcal{E}^*$ is similar. For each $A \in {}_\Lambda\mathfrak{M}$ we consider the Γ-module ${}^{(\varphi)}A = \mathrm{Hom}_\Lambda(\Gamma, A)$ and prove that it is $\mathcal{E}$-injective. Indeed

$$\mathrm{Hom}_\Gamma(B, {}^{(\varphi)}A) = \mathrm{Hom}_\Gamma(B, \mathrm{Hom}_\Lambda(\Gamma, A))$$

$$\approx \mathrm{Hom}_\Lambda(\Gamma \otimes_\Gamma B, A) \approx \mathrm{Hom}_\Lambda(B, A).$$

Thus if B is an exact sequence in $\mathcal{E}$ then B Λ-splits and $\mathrm{Hom}_\Lambda(B,A)$ is exact (II,5, Prop.1). Thus $\mathrm{Hom}_\Gamma(B, {}^{(\varphi)}A)$ is exact and thus ${}^{(\varphi)}A$ is $\mathcal{E}$-injective. The balance of the argument is similar.