RÉDACTION N° 171TER

TITRE : THÉORIE DES SÉRIES L

ASSOCIATION DES COLLABORATEURS DE NICOLAS BOURBAKI

NOMBRE DE PAGES : 75

NOMBRE DE FEUILLES : 75

THE INSTITUTE FOR ADVANCED STUDY 171 ter

Princeton - New-Jersey

K. IWASAWA.- Théorie des Séries L. - April 1952

Let k be a finite algebraic number field, J the idele group of k, topologized as in a recent paper of Weil. J is a locally compact abelian group containing the principal idèle group $\mathbb{P}$ as a discrete subgroup. We denote by J_0 the subgroup of J consisting of idèles $\mathfrak{a} = (a_p)$ such that $a_p=1$ for all infinite (i.e. archimedean) primes P. We call J_0 the finite part of J and define the infinite part J_∞ similarly, so that we have

$$J = J_0 \times J_\infty \quad , \quad \mathfrak{a} = \mathfrak{a}_0 \mathfrak{a}_\infty , \quad \mathfrak{a}_0 \in J_0 , \quad \mathfrak{a}_\infty \in J_\infty .$$

We also denote by U the compact subgroup of J consisting of ideles $\mathfrak{a} = (a_p)$ such that the absolute value $|a_p|_p = 1$ for every prime P. $U_0 = U \cap J_0$ is then an open, compact subgroup of J_0 and J_0/U_0 is canonically isomorphic to the ideal group I of k. According to Artin-Whaples, we can choose the absolute values $|a_p|_p$ so that the volume function $V(\mathfrak{a}) = \prod_p |a_p|_p$ $(\mathfrak{a} = (a_p))$ has the value 1 at every principal idèle $\alpha \in \mathbb{P}$ (the product formula) and that $V(\mathfrak{a}_0)^{-1}$ is equal to the absolute norm $N(\tilde{\mathfrak{a}}_0)$ of the ideal $\tilde{\mathfrak{a}}_0$, which corresponds to $\mathfrak{a}_0$ by the above isomorphism between J_0/U_0 and I.

We now define a function $\varphi(\mathfrak{a})$ by

$$\varphi(\mathfrak{a}) = \varphi(\mathfrak{a}_0)\,\varphi(\mathfrak{a}_\infty), \qquad \mathfrak{a} = \mathfrak{a}_0 \mathfrak{a}_\infty ,$$

$$\varphi(\mathfrak{a}_0) = \begin{cases} 1 , & \text{if } \tilde{\mathfrak{a}}_0 \text{ is an integral ideal,} \\ 0 , & \text{otherwise,} \end{cases}$$

3

$$\varphi(\mathfrak{a}_\infty) = \exp\left(-\frac{\pi}{\sqrt[n]{\Delta}} \sum_{i=1}^{r} e_i |a_{p_{\infty,i}}|^2\right),$$

where n is the absolute degree of k , Δ is the discriminant of k , $a_{p_{\infty,i}}$ are the components of $\mathfrak{a}$ at the infinite primes $P_{\infty,i}$ and $e_o=1$ or 2 according as $P_{\infty,i}$ is real or complex. Since U_0 is open in J_0, $\varphi(\mathfrak{a})$ is a continuous function on J and we define a function $\xi(s)$ by

(1) $$\xi(s) = \int_J \varphi(\mathfrak{a}) V(\mathfrak{a})^s d\mu(\mathfrak{a}), \quad \text{for} \quad s > 1 .$$

Here $\mu(\mathfrak{a})$ denotes a Haar measure of the locally compact group J . We shall calculate this integral in two different ways.

First, using $J = J_0 \times J_\infty$, $\varphi(\mathfrak{a}) = \varphi(\mathfrak{a}_0)\varphi(\mathfrak{a}_\infty)$ and $V(\mathfrak{a}) = V(\mathfrak{a}_0)V(\mathfrak{a}_\infty)$, we have

$$\xi(s) = \int_{J_0} \varphi(\mathfrak{a}_0)V(\mathfrak{a}_0)^3 d\mu(\mathfrak{a}_0) \int_{J_\infty} \varphi(\mathfrak{a}_\infty)V(\mathfrak{a}_\infty)^s d\mu(\mathfrak{a}_\infty) .$$

If we note that U_0 is an open, compact subgroup of J_0 and $J_0/U_0 \cong I$, we see immediately that the first integral on the right-hand side is equal to (up to a positive constant) the zeta-function $\zeta(s) = \sum N(\tilde{\mathfrak{a}})^{-s}$ ($\tilde{\mathfrak{a}}$ = integral ideal) of k . On the other hand, J_∞ being the direct product of r copies of the multiplicative group K^* of the real or complex number-field K , the second integral is the product of integrals of the form

$$\int_{K^*} \exp\left(-\frac{\pi}{\sqrt[n]{\Delta}} e|t|^2\right)|t|^s d\mu_K(t) , \qquad e = 1 \text{ or } 2 ,$$

which can be easily calculated to be equal to

$$\Delta^{\frac{s}{2n}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \quad \text{or} \quad \Delta^{\frac{s}{n}} 2^{-s} \pi^{-s} \Gamma(s) ,$$

according as K is real or complex. We have therefore

(2) $$\xi(s) = \text{const.}\ 2^{-r_2 s} \Delta^{\frac{s}{2}} \pi^{-\frac{r_1 s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta(s) .$$

The above calculation also shows that the integral (1) actually converges for $s > 1$.

We now transform the same integral (1) in another way. Namely, we first integrate the function $f(\mathfrak{a})=\varphi(\mathfrak{a})V(\mathfrak{a})^3$ on the subgroup $\mathbb{P}$ and then on the factor group $\bar{J} = J/\mathbb{P} = \{\bar{\mathfrak{a}}\}$;

$$\int_J f(\mathfrak{a})d\mu(\mathfrak{a}) = \int_{\bar{J}} \left\{ \int_{\mathbb{P}} f(\mathfrak{a}\alpha)d\mu(\alpha) \right\} d\mu(\bar{\mathfrak{a}}) .$$

However, since P is discrete and $V(\mathfrak{a}\alpha) = V(\mathfrak{a})V(\alpha) = V(\mathfrak{a})=V(\bar{\mathfrak{a}})$, we have

$$\int_{\mathbb{P}} f(\mathfrak{a}\alpha)d\mu(\alpha) = \Big(\sum_{\alpha\in P} \varphi(\mathfrak{a}\alpha)\Big)V(\bar{\mathfrak{a}})^s ,$$

and if we put

$$\bar{\varphi}(\bar{\mathfrak{a}}) = \sum_{\alpha\in P} \varphi(\mathfrak{a}\alpha) ,$$

$$\Theta(\bar{\mathfrak{a}}) = 1 + \bar{\varphi}(\bar{\mathfrak{a}}) = \sum_{\alpha\in k} \varphi(\mathfrak{a}\alpha) ,$$

the theta-formula

$$\Theta(\bar{\mathfrak{a}}) = V(\bar{\mathfrak{a}})^{-1}\Theta(\bar{\vartheta}\bar{\mathfrak{a}}^{-1}) \quad \text{or} \quad \bar{\varphi}(\bar{\mathfrak{a}}) = V(\bar{\mathfrak{a}})^{-1}\bar{\varphi}(\bar{\vartheta}\bar{\mathfrak{a}}^{-1})+V(\bar{\mathfrak{a}})^{-1} -1$$

holds, where ϑ is an idèle of volume 1 such that $\tilde{\vartheta}_0$ is the different of k and its infinite components are all equal to $\sqrt[n]{\Delta}$. We have now

$$\xi(s) = \int_{\bar{J}} \bar{\varphi}(\bar{\mathfrak{a}})V(\bar{\mathfrak{a}})^3 d\mu(\bar{\mathfrak{a}}) = \int_{V(\bar{\mathfrak{a}})\geqq 1} + \int_{V(\bar{\mathfrak{a}})\leqq 1} ,$$

and here the first integral on the right-hand side

$$\psi(s) = \int_{V(\bar{\mathfrak{a}})\geqq 1} \bar{\varphi}(\bar{\mathfrak{a}})V(\bar{\mathfrak{a}})^s d\mu(\bar{\mathfrak{a}})$$

gives an integral function of s , for this integral converges absolutely for every complex value s , because of the convergence of (1) for $s>1$ and because of $V(\bar{\mathfrak{a}})\geqq 1$. Using the theta-formula and the invariance of Haar measures, we can transform the second integral as follows :

$$\int_{V(\bar{\mathfrak{a}})\leqq 1} = \int_{V(\bar{\mathfrak{a}})\leqq 1} (V(\bar{\mathfrak{a}})^{-1}\bar{\varphi}(\bar{\vartheta}\bar{\mathfrak{a}}^{-1})+V(\bar{\mathfrak{a}})^{-1}-1)V(\bar{\mathfrak{a}})^s d\mu(\bar{\mathfrak{a}})$$

$$= \int_{V(\bar{\mathfrak{a}})\geqq 1} (\bar{\varphi}(\bar{\vartheta}\bar{\mathfrak{a}})V(\bar{\mathfrak{a}})^{1-s} + V(\bar{\mathfrak{a}})^{1-s} - V(\bar{\mathfrak{a}})^{-s})d\mu(\bar{\mathfrak{a}}) \quad (\text{by } \bar{\mathfrak{a}} \to \bar{\mathfrak{a}}^{-1})$$

$$= \int_{V(\bar{\mathfrak{a}})\geqq 1} \bar{\varphi}(\bar{\vartheta}\bar{\mathfrak{a}})V(\bar{\mathfrak{a}})^{1-s}d\mu(\bar{\mathfrak{a}}) + \int_{V(\bar{\mathfrak{a}})\geqq 1} (V(\bar{\mathfrak{a}})^{1-s} - V(\bar{\mathfrak{a}})^{-s})d\mu(\bar{\mathfrak{a}})$$

$$= \int_{V(\bar{\mathfrak{a}})\geqq 1} \bar{\varphi}(\bar{\mathfrak{a}})V(\bar{\mathfrak{a}})^{1-s}d\mu(\bar{\mathfrak{a}}) + \int_{V(\bar{\mathfrak{a}})\geqq 1} (V(\bar{\mathfrak{a}})^{1-s} - V(\bar{\mathfrak{a}})^{-s})d\mu(\bar{\mathfrak{a}})$$

$$(\text{by } \bar{\mathfrak{a}} \to \bar{\vartheta}^{-1}\bar{\mathfrak{a}} \quad \text{and} \quad V(\bar{\vartheta}) = 1)$$

$$= \psi(1-s) + \int_{V(\bar{\mathfrak{a}})\geqq 1} (V(\bar{\mathfrak{a}})^{1-s} - V(\bar{\mathfrak{a}})^{-s})d\mu(\bar{\mathfrak{a}}) .$$

Now, the set of all ideles $\mathfrak{A}$ such that $V(\mathfrak{A})=1$ forms a closed subgroup J_1 of J and it can be seen easely that J is the direct product of $\overline{J}_1 = J_1/P$ and a subgroup S which is canonically isomorphic to the multiplicative group $T = \{t = V(\overline{\mathfrak{A}})\}$ of positive real numbers. Hence we have

$$\int_{V(\overline{\mathfrak{A}}) \geqq 1} (V(\overline{\mathfrak{A}})^{1-s} - V(\overline{\mathfrak{A}})^{-s} d\mu(\overline{\mathfrak{A}}) = \int_{\overline{J}_1} \times \int_{S, V(\overline{\mathfrak{A}}) \geqq 1}$$

$$= \mu(\overline{J}_1) \int_{t \geqq 1} (t^{1-s} - t^{-s}) \frac{dt}{t}$$

$$= \mu(\overline{J}_1)\left(\frac{1}{s-1} - \frac{1}{s}\right).$$

We have, therefore, the formula

$$(3) \qquad \xi(s) = \psi(s) + \psi(1-s) + \mu(\overline{J}_1)\left(\frac{1}{s-1} - \frac{1}{s}\right), \qquad (s > 1).$$

It then follows immediately that $\xi(s)$ is a regular analytic function of s on the whole s-plane except for simple poles at $s=0, 1$ and it satisfies the equation

$$\xi(s) = \xi(1-s),$$

which is nothing but the functional equation of the zeta-function $\xi(s)$ (cf. (2)).

The formula (3) also shows that the measure $\mu(\overline{J}_1)$ of $\overline{J}_1$ is finite. Since $\overline{J}_1$ is a locally compact group, this means that $\overline{J}_1$ is compact. Now, we put $H = (U_0 \times J_\infty) \cap J_1$ and consider the sequence of groups

$$J_1 \supset HP \supset UP \supset P .$$

Since U is compact UP is closed in J_1, and, since $U_0 \times J_\infty$ is open in J, H and HP are open subgroupe of J_1. It then follows from the compactness of $\overline{J}_1 = J_1/P$ that J_1/HP and HP/UP are both compact groups. But, as HP is open and J_1/HP is discrete, J_1/HP must be finite. Consequently, the group $J/(U_0 \times J_\infty)P$, which is easily seen to be isomorphic to J_1/HP, is a finite group and this proves the finiteness of the ideal classes of k. Now, H/U is isomorphic to $(J_1 \cap J_\infty)/(U \cap J_\infty)$ and hence is an $(r-1)$-dimensional vector group. On the other hand, we see from the isomorphisms

$$HP/UP = H/U(H \cap P), \qquad U(H \cap P)/U = H \cap P/U \cap P ,$$

that $H/U(H \cap P)$ is compact and $U(H \cap P)/U$ is discrete. Since H/U is a vector group, this implies that $U(H \cap P)/U$ is an $(r-1)$-dimensional lattice in H/U and, consequently, that $H \cap P/U \cap P$ is a free abelian group with $r-1$ generators. However, as is readily seen, $H \cap P$ and $U \cap P$ are the unit group and the group of roots of unity in k. Hence the classical Dirichlet's unit theorem has been proved.

The above method of proving the functional equation can be also applied to Hecke's L-functions with "Grossencharakteren", for such a character χ is a continuous character of $\bar{J}$ which is trivial on S. The integrand of (1) must be then replaced by

$$\chi(\alpha)\, \varphi(\alpha, \chi) V(\alpha)^s ,$$

where $\varphi(\alpha, \chi)$ is a similar function to $\varphi(\alpha)$, depending on χ. The zeta-function (or L-functions) of a division algebra over a finite algebraic number-field can also be treated in a similar way, though here integrations over linear groups appear and calculations are more complicated.

For the above proof of the functional equation of $\zeta(s)$, two group-theoretical facts seem to be essential. One is the topological structure of the group J, that of its subgroups and factor groups, together with the invariance of Haar measures on them, and the other is the theta-formula, which is an analytical expression for the self-duality of the additive group of the ring R of valuation vectors (= additive idèles) of k. J being exactly the multiplicative group of R, here the additive and multiplicative properties of R are subtly mixed up and it seems to me likely that something essential to the arithmetic of k is still hidden in this connection, though I only know that the usual topology of J coincides with the one which is obtained by considering J as a group of automorphisms of the additive group of R in the sense of Braconnier.
