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# Translation invariance, compact semigroups and injective algebras


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# Translation invariance, compact semigroups and injective coalgebras

by Andrew TONGE

L'absence d'une notion satisfaisante du dual d'un semigroupe compact S empêche une caractérisation concrète et simple des sous-espaces fermés de C(S) invariants par translation. Dans cet article nous développons une théorie abstraite de tels espaces (satisfaisants à une version faible de la propriété d'approximation de Banach) en termes de "coalgèbres injectives". Nous donnons une caractérisation de leurs espaces duaux comme les algèbres de Banach dont les boules unitées munies de la topologie faible étoile sont des semigroupes compacts. On identifie les algèbres de Hopf injectives aux espaces C(G), où G est un groupe compact.

The lack of a satisfactory notion of the dual of a compact semigroup S precludes a simple concrete characterisation of the closed translation invariant subspaces of C(S). In this article, we develop an abstract theory of such spaces (satisfying a weak version of the Banach approximation property) in terms of "injective coalgebras". We characterise their dual spaces as those Banach algebras whose closed unit balls are compact semigroups under the weak star topology. Injective Hopf algebras are shown to be the spaces C(G), G a compact group.

## 1. INTRODUCTION.

Let $G$ be a compact abelian group and let $\Gamma$ be its dual. Then all closed translation invariant subspaces of $C(G)$, the Banach space of all continuous (complex valued) functions on $G$, under the uniform norm, have the form

(*) $$C_\Lambda(G) = \left\{ f \in C(G) : \hat{f}(\gamma) = 0 \quad \text{if} \quad \gamma \notin \Lambda \right\}$$

for some $\Lambda \subseteq \Gamma$. Conversely, all such spaces are translation invariant. Using the representation theory for compact groups, one may extend this result to the non-abelian case, but there is no analogous characterisation of the closed translation invariant subspaces of $C(S)$, $S$ a compact semigroup (with jointly continuous multiplication). This is essentially because there is in general an insufficient supply of semicharacters. As a simple example, consider $S = [0,1]$ with the usual topology and with multiplication

$$s.t = \max(s,t).$$

This compact semigroup has no non-trivial semicharacters. Thus

$\left\{ f \in C(S) : f(s) = 0 \quad \text{if} \quad s \geq \frac{1}{2} \right\}$ can have no interpretation like (*). It is, nevertheless, a closed translation invariant subspace of $C(S)$.

Although a concrete characterisation of the closed translation invariant subspaces of $C(S)$, $S$ a compact semigroup, appears elusive, we are able to give an abstract characterisation in terms of "injective coalgebras" (defined in § 2) provided that we assume a weak version of the Banach approximation property.

The Banach space $B$ is said to have the Banach approximation property (BAP for short) if, in the topology of compact convergence, the identity operator on $B$ is a limit of finite rank bounded linear operators on $B$. We shall work with an alternative formulation, due to Grothendieck, which uses the language of tensor products.

Suppose that $X$ and $Y$ are Banach spaces, and write $(X')_1$ for the closed unit ball of the dual $X'$ of $X$, etc... The injective tensor product $X \check{\otimes} Y$ is defined to be the completion of $X \otimes Y$ under the following norm :

$$\|\tau\|_{X \check{\otimes} Y} = \sup\left\{ |\langle \tau, x' \otimes y' \rangle| : x' \in (X')_1 , y' \in (Y')_1 \right\} \quad \text{if} \quad \tau \in X \otimes Y.$$

$\check{\otimes}$ is the smallest of Grothendieck's natural $\otimes$-norms [3].

Let $L(X',Y)$ denote the usual Banach space of bounded linear mappings $X' \to Y$. We shall say that $u \in L(X',Y)$ is weakly continuous if it is continuous from $(X', \sigma(X',X))$ to $(Y, \sigma(Y,Y'))$. Write $FW(X',Y)$ for the

uniform closure of the weakly continuous elements of $L(X',Y)$ of finite rank ; write $KW(X',Y)$ for the closed subspace of $L(X',Y)$ consisting of the compact weakly continuous mappings.

Now $X \check{\otimes} Y$ may be (and will be) identified naturally with $FW(X',Y) \subseteq KW(X',Y)$.

CRITERION (Grothendieck [2]). The Banach space $B$ as the BAP iff $KW(X',B) = X \check{\otimes} B$ for every Banach space $X$.

We shall use a (probably) weaker condition.

DEFINITION. The Banach space $B$ <u>has the bap</u> iff $KW(B',B) = B \check{\otimes} B$.

The classical Banach spaces have the BAP. However, Enflo constructed a Banach space for which the BAP fails (see [1] for a simplified version) and from this we can deduce that there are Banach spaces which do not even have the bap. For, let $B$ be a Banach space which does not have the BAP. Then there is a Banach space $X$ for which $KW(X',B) \supsetneqq FW(X',B)$. We claim that $A = X \oplus B$ (the $\ell^2$ direct sum) does not have the bap. If $u \in KW(X',B) \setminus FW(X',B)$, define $U : X' \oplus B' \to X \oplus B$ by $U(x'+b') = u(x')$ $\forall x' \in X'$, $\forall b' \in B$. Then it is easy to check that $U \in KW(A',A)$. On the other hand $U \notin FW(A',A)$, for if it were, we could find weakly continuous $F_n \in L(A',A)$ of finite rank such that $\lim_{n\to\infty} \|F_n - U\| = 0$. But, if $\Pi$ is the projection $A \to B$ and $J$ is the natural inclusion $X' \to A'$, then $u = \Pi U J$. Thus $\lim_{n\to\infty} \|\Pi F_n J - u\| = 0$, a contradiction. This establishes the claim.

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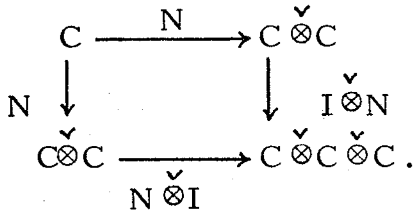
## 2. INJECTIVE COALGEBRAS.

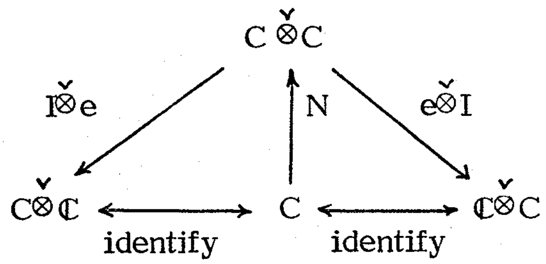
Banach algebras may be defined in terms of topological tensor products and commuting diagrams. To do this, we interpret the multiplication on the Banach algebra $R$ as a linear contraction $M : R \hat{\otimes} R \to R$ (where $R \otimes R$ is the completed projective tensor product of $R$ with itself [2]). It is now natural to consider what happens if the arrows in the diagrams are reversed. The richest theory is obtained if we also replace $\hat{\otimes}$ by $\check{\otimes}$. The structures obtained in this way are called coalgebras The purely algebraic theory has been investigated extensively - a good source is [6] - and has applications in Galois theory and in the theory of Lie groups. Where our results are essentially algebraic, we shall be very brief.

DEFINITION. Suppose that $C$ is a Banach space and that $N : C \to C \check{\otimes} C$ is a linear contraction. Then the pair $(C,N)$ – or simply $C$ – is said to be an <u>injective coalgebra</u> if the diagram commutes




Here, and elsewhere, $I$ denotes the identity endomorphism. $N$ is called the <u>comultiplication</u> and the commutativity of the diagram is referred to as <u>coassociativity</u>.

If there is a linear map $e : C \to \mathbb{C}$ of norm 1 such that




commutes, then $(C,N,e)$ – or simply $C$ – is said to be a <u>counital</u> <u>injective coalgebra</u>. $e$ is called the <u>coidentity</u>.

Let $T : C \check{\otimes} C \to C \check{\otimes} C$ be the linear isometry generated by $T(c \otimes d) = d \otimes c$ $\forall c,d \in C$. Henceforth, the letter $T$ is reserved for reflection mappings of this form. $C$ is said to be <u>cocommutative</u> if $T \circ N = N$.

Many well known Banach spaces have a natural injective coalgebra structure. The most important is $C(S)$, $S$ a compact semigroup. If we identify $C(S) \check{\otimes} C(S)$ with $C(S \times S)$, the comultiplication is given by

$$(Nf)(s,t) = f(st) \qquad \forall s, t \in S \quad \forall f \in C(S).$$

If $S$ is abelian, then $C(S)$ is cocommutative. If $S$ has an identity, then evaluation at the identity provides a coidentity for $C(S)$.

Other examples are

(1) $L^p(\mathbf{T})$ $(1 \leq p < \infty)$. The comultiplication is generated by

$$N\chi_n = \chi_n \otimes \chi_n \qquad (n \in \mathbf{Z}).$$

Here, and later, $\chi_n(t) = \exp(int)$ $(t \in \mathbf{T})$.

(2) $\ell^p$ $(1 \leq p < \infty)$. The comultiplication is generated by

$$N\, e_n = e_n \otimes e_n \qquad (n \in \mathbf{Z})$$

where the $e_n$ are the usual co-ordinate vectors.

(3) Any Banach space $B$ may be given a trivial injective coalgebra structure. If $b_o \in B$, $\|b_o\| \leq 1$, define $Nb = b \otimes b_o$ $\forall b \in B$.

Just as a Banach algebra may always be embedded in a unital Banach algebra, so we may adjoin a coidentity to an injective coalgebra. Suppose that $(C,N)$ is an injective coalgebra. Define $C_1 = C \oplus \mathbb{C}$ (the $\ell^\infty$ direct sum). If

$$N_1(c+\lambda) = Nc + c \otimes 1 + 1 \otimes c + \lambda 1 \otimes 1$$

and $$e_1(c+\lambda) = \lambda, \quad \forall c \in C, \ \forall \lambda \in \mathbb{C},$$

then $(C_1, N_1, e_1)$ is a counital injective coalgebra.

The following lemma will be important later.

LEMMA 2.1. Suppose that $(C, N_C, (e_C))$ and $(D, N_D, (e_D))$ are (counital) injective coalgebras. Then $C \check{\otimes} D$ is canonically a (counital) injective coalgebra.

Proof. The comultiplication is given by the composite

$$C \check{\otimes} D \xrightarrow[N_C \check{\otimes} N_D]{} C \check{\otimes} C \check{\otimes} D \check{\otimes} D \xrightarrow[I \check{\otimes} T \check{\otimes} I]{} C \check{\otimes} D \check{\otimes} C \check{\otimes} D$$

where $T$ is the reflection mapping.

The coidentity is $e_C \check{\otimes} e_D$ (identifying $\mathbb{C} \check{\otimes} \mathbb{C}$ with $\mathbb{C}$).

A simple result of great significance is

PROPOSITION 2.2. Suppose that $C$ is an injective coalgebra. Then $C'$ has a canonical Banach algebra structure. $C'$ is unital iff $C$ is counital. $C'$ is commutative iff $C$ is cocommutative.

Proof. If $x,y \in C'$ we define $xy \in C'$ by

$$\langle c,xy\rangle = \langle Nc , x\otimes y\rangle \quad \forall c\in C.$$

If $C$ has coidentity $e$, then $e$ is the identity of $C$.

Unfortunately, the dual of a Banach algebra is not necessarily an injective coalgebra. However, a satisfactory substitute involving concepts of almost periodicity may be constructed. [7].

In this situation, it is natural to ask which elements of an injective coalgebra $C$ are multiplicative linear functionals on $C'$.

LEMMA 2.3. The $\sigma(C',C)$-continuous multiplicative linear functionals on $C'$ are precisely those elements $a$ of $C$ for which $Na = a\otimes a$. We call such (non-zero) elements atoms.

Let us consider some examples.

(1) If $S$ is a compact semigroup, then the injective coalgebra structure on $C(S)$ induces the convolution multiplication on $C(S)' = M(S)$. The atoms of $C(S)$ are the semicharacters of $S$.

(2) The injective coalgebra structure on $L^p(\mathbf{T})$ considered above induces convolution multiplication on $L^{p'}(\mathbf{T})$ $(\frac{1}{p}+\frac{1}{p'} = 1,\ \ 1 \le p < \infty)$. The atoms of $L^p(T)$ are the functions $\chi_n$ $(n\in\mathbf{Z})$.

(3) The injective coalgebra structure on $\ell^p$ considered above induces pointwise multiplication on $\ell^{p'}$ $(\frac{1}{p}+\frac{1}{p'}=1, \quad 1 \le p < \infty)$. The atoms of $\ell^p$ are the coordinate functions $e_n$ $(n \in \mathbf{Z})$.

It is well known that if $R$ is a Banach algebra, then $R'$ is naturally a left or right normed R-module.

PROPOSITION 2.4. Suppose that $C$ is an injective coalgebra. Then $C$ is a left normed $C'$-module.

Proof. Fix $r \in C'$ and $c \in C$. Define $r \rhd c \in C''$ by

$$\langle r \rhd c\, ,\, s\rangle = \langle c\, ,\, rs\rangle \qquad \forall s \in C'.$$

We show that $r \rhd c$ is a norm limit of elements of $C$, and so is itself in $C$. Fix then $\varepsilon > 0$ and choose $\Sigma_{j=1}^{J} c_j \otimes d_j \in C \otimes C$ with $\|Nc - \Sigma_{j=1}^{J} c_j \otimes d_j\|_{C \check{\otimes} C} < \varepsilon$. Then $\|r \rhd c - \Sigma_{j=1}^{J} \langle c_j\, ,\, r\rangle d_j\|_{C''} = \sup\{|\langle r \rhd c, s\rangle - \langle \Sigma_{j=1}^{J} c_j \otimes d_j, r \otimes s\rangle| : s \in (C')_1\}$ $\le \varepsilon\|r\|$, and we deduce that $r \rhd c \in C$. The rest is immediate.

Similarly, $C$ is a right normed $C'$-module with action $c \lhd r$ defined by $\langle c \lhd r\, ,\, s\rangle = \langle c\, ,\, sr\rangle \quad \forall c \in C, \quad \forall r,s \in C'$.

We are now in a position to prove our first main result, which gives a manageable way of discovering when a Banach algebra is the dual of an injective coalgebra.

THEOREM 2.5. Let $R$ be a Banach algebra with unit ball $X$. Suppose that $R$ has a predual $C$ and consider $X$ as a compact Hausdorff space under the $\sigma(R,C)$ topology.

(i) <u>If the Banach algebra structure on</u> $R$ <u>is induced (as in 2.2) by an injective coalgebra structure on</u> $C$, <u>then</u> $X$ <u>is a compact semigroup</u>.

(ii) <u>If</u> $C$ <u>has the bap and if</u> $X$ <u>is a compact semigroup, then</u> $C$ <u>may be given an injective coalgebra structure which induces the Banach algebra structure on</u> $R$.

$X$ <u>will be unital, resp. abelian iff</u> $C$ <u>is counital, resp. cocommutative</u>.

<u>Proof</u>. (i) $X$ is clearly a semigroup. We must show that the multiplication is jointly $\sigma(R,C)$ continuous. Suppose that $x_\alpha \to x$ and $y_\beta \to y$ in $(X,\sigma(R,C))$. Fix $\varepsilon > 0$ and $c \in C$. Then

$$\langle c, x_\alpha y_\beta - xy\rangle = \langle Nc, (x_\alpha - x)\otimes(y_\beta - y)\rangle + \langle x \triangleright c\ ,\ y_\beta - y\rangle + \langle c \triangleleft y\ ,\ x_\alpha - x\rangle .$$

Now $\exists\ \alpha_0\ ,\ \beta_0$ such that $\alpha \geq \alpha_0 \Rightarrow |\langle c \triangleleft y\ ,\ x_\alpha - x\rangle| < \varepsilon/3$ and $\beta \geq \beta_0 \Longrightarrow |\langle x \triangleright c\ ,\ y_\beta - y\rangle| < \varepsilon/3$. Choosing $\Sigma_{j=1}^{J} c_j \otimes d_j \in C \otimes C$ with $\|Nc - \Sigma_{j=1}^{J} c_j \otimes d_j\|_{C \check{\otimes} C} < \varepsilon/24$, we can find $\alpha_1 \geq \alpha_0$ and $\beta_1 \geq \beta_0$ such that for $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$ we have

$$|\langle Nc\ ,\ (x_\alpha - x)\otimes(y_\beta - y)\rangle| < \varepsilon/6 + \Sigma_{j=1}^{J} |\langle c_j\ ,\ x_\alpha - x\rangle\langle d_j\ ,\ y_\beta - y\rangle|$$
$$< \varepsilon/3,$$

whence $|\langle c\ ,\ x_\alpha y_\beta - x_y\rangle| < \varepsilon$. Consequently $x_\alpha y_\beta \to xy$ in $(X\ ,\ \sigma(R,C))$.

(ii) Fix $c \in C$ and $r \in R$ and define $f(c,r) \in C''$ by $\langle s, f(c,r)\rangle = \langle c, rs\rangle$ $\forall s \in R$. One may verify that $f(c,r)|_X$ is $\sigma(R,C)$-continuous, whence $f(c,r) \in C$. Now, allowing $r$ to vary, we define a linear mapping $(Nc) : C' \to C$ by $(Nc)(r) = f(c,r)$. It is easy to check that $(Nc)$ is weakly continuous. Since $C$ has the bap, we can identify $(Nc)$ with an element of $C \check{\otimes} C$ if we can show it to be a compact mapping. This will follow if $(Nc)|_X$ is continuous $(R\ ,\ \sigma(R,C))$ to $(C\ ,\ \|\cdot\|)$.

Thus, given $\varepsilon > 0$, we must find a $\sigma(R,C)$ neighbourhood of $0 \in R - U$, say - such that $s \in U \cap X \Longrightarrow \|(Nc)(s)\| < \varepsilon$.

By hypothesis, there is an absolutely convex $\sigma(R,C)$ neighbourhood $V$ of $0 \in R$ such that

$$u,v \in V \cap X \Longrightarrow |\langle c,uv\rangle| < \varepsilon/4.$$

As $X$ is $\sigma(R,C)$ compact there is a finite set $\{r_1,\ldots,r_N\} \subset X$ such that $X \subseteq \bigcup_{n=1}^{N} (V + r_n)$. Now, in the procedure above, we might just as well have defined a mapping $(Иc) : C' \to C$ by $\langle (Иc)(x),y\rangle = \langle c\ ,\ yx\rangle$ $\forall x,y \in R$. The properties of $(Иc)$ will be similar to those of $(Nc)$. Consider

$$V_n = \left\{y\in R : |\langle (Иc)(r_n),y\rangle| < \varepsilon/2\right\} \qquad (1 \le n \le N).$$

Then $U = V \cap (\bigcap_{n=1} V_n)$ is a $\sigma(R,C)$ neighbourhood of $0 \in R$. Choose $s \in U \cap X$. Then $|\langle c,sr_n\rangle| < \varepsilon/2$ $(1 \le n \le N)$. Taking $r\in X$, $\exists\, i \le \nu \le N$ such that $r \in V + r_\nu$. Then $(r - r_\nu)/2 \in V \cap X$, so $|\langle c,s(r-r_\nu)\rangle| < \varepsilon/2$. Adding, we obtain $|\langle (Nc)(s),r\rangle < \varepsilon$, and since this holds $\forall r \in X$ we have $\|(Nc)(s)\| < s$, as required.

We now have a mapping $N : C \to C \check{\otimes} C$ ; $c \mapsto (Nc)$, which clearly defines an appropriate comultiplication.

As illustrations, it can be shown that

(1) The Banach algebra $L^\infty(\mathbf{T})$ under pointwise multiplication is not the dual of an injective coalgebra.

(2) If $H_0^1(\mathbf{T}) = \left\{f\in L^1(\mathbf{T}) : \hat{f}(n) = 0 \quad (n \le 0)\right\}$ and $H^\infty(\mathbf{T}) = \left\{f\in L^\infty(\mathbf{T}) : \hat{f}(n) = 0 \quad (n < 0)\right\}$, then there is an injective coalgebra structure on $L^1(\mathbf{T})/H_0^1(\mathbf{T})$ which induces the Banach algebra structure on its dual $H^\infty(\mathbf{T})$

given by pointwise multiplication.

This stands in contrast with the situation for convolution multiplication on $L^{\infty}(\mathbf{T})$ and $H^{\infty}(\mathbf{T})$.

Our next objective is the characterisation of injective coalgebras as closed translation invariant subspaces of $C(S)$, $S$ a compact semigroup. The precise result is

THEOREM 2.6. (i) An injective coalgebra is a closed translation invariant subspace of $C(S)$, $S$ some compact semigroup.

(ii) If a closed translation invariant subspace of $C(S)$, $S$ a compact semigroup, has the bap, then it is an injective coalgebra under the comultiplication induced from $C(S)$.

One may add the usual statements about identities and commutativity.

To see that a closed translation invariant subspace of $C(S)$ need not have the bap, simply take an injective coalgebra which does not have the bap and apply 2.6 (i).

To prove 2.6, it is natural to talk about subcoalgebras and coideals. This raises some unexpected difficulties which we discuss below. Working on the principle that arrows are created to be reversed, the following definition is very natural.

DEFINITION. Let $J$ be a closed subspace of the injective coalgebra $C$

(i) $J$ is a natural left (right) coideal if

$$N : J \to C \check{\otimes} J \qquad (J \to J \check{\otimes} C).$$

(ii) $J$ is a natural coideal if $N : J \to \overline{J \otimes C + C \otimes J}$ (closure in $C \check{\otimes} C$), and (when $C$ is counital) $e(J) = 0$.

(iii) $J$ is a <u>natural subcoalgebra</u> if $N : J \to J \check{\otimes} J$.

On the other hand to justify the term "coideal" we must be able to construct quotients. More precisely, coideals should be kernels of structure preserving mappings.

DEFINITION. Suppose that $C$ and $D$ are injective coalgebras and that $\varphi : C \to D$ is a linear contraction. Then $\varphi$ is called a <u>coalgebra contraction</u> if the diagram below commutes

$$\begin{array}{ccc} C & \xrightarrow{\quad N \quad} & C \check{\otimes} C \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle \varphi \check{\otimes} \varphi} \\ D & \xrightarrow[\quad N \quad]{} & D \check{\otimes} D \end{array}$$

If $C$ and $D$ are counital, we also demand that $e_D \circ \varphi = e_C$.

It is easy to verify that the composition and injective tensor product of two coalgebra contractions is again a coalgebra contraction. Using the fact that if $\varphi : C \to D$ is a coalgebra contraction, then ${}^t\varphi : D' \to c'$ is an algebra contraction, one may show that if $J = \ker \varphi$ then $J \lhd J^o \subset J$ (and, equivalently $J^o \rhd J \subseteq J$) where $J^o$ denotes the annihilator of $J$ in $C'$. If $C$ and $D$ are counital, then $e_C(J) = 0$. One may quickly verify

PROPOSITION 2.7. <u>Suppose that</u> $J$ <u>is a closed subspace of the injective coalgebra</u> $C$. <u>Then</u> $J^o$ <u>is a weak* closed subspace of</u> $C'$, <u>and</u>

(i) $C' \rhd J \subseteq J \Leftrightarrow J^o$ <u>is a left ideal of</u> $C'$

(ii) $J \lhd C' \subseteq J \Leftrightarrow J^o$ <u>is a right ideal of</u> $C'$

(iii) $J \lhd J^o \subseteq J \Leftrightarrow J^o$ <u>is a subalgebra of</u> $C'$.

<u>If</u> $C$ <u>is counital, we have</u>

(iii)' $J \lhd J^o \subseteq J$ <u>and</u> $e(J) = 0 \Leftrightarrow J^o$ <u>is a unital subalgebra of</u> $C'$.

With this motivation we make the following.

DEFINITION. Let $J$ be a closed subspace of the injective coalgebra $C$.

(i) $J$ is a <u>left (right) coideal</u> iff $C' \rhd J \subseteq J$ $(J \lhd C' \subseteq J)$.

(ii) $J$ is a <u>subcoalgebra</u> iff $C' \rhd J \subseteq J$ and $J \lhd C' \subseteq J$.

(iii) $J$ is a <u>coideal</u> iff $J \lhd J^{o} \subseteq J$ (and, if $C$ is counital, $e(J) = 0$).

Then, we see that the kernel of a coalgebra contraction is a coideal, and that if $J$ is a coideal of $C$ and $\pi : C \to C/J$ is the quotient map, then $C/J$ has a unique injective coalgebra structure with respect to which $\pi$ is a coalgebra contraction. Also, the intersection of subcoalgebras is again a subcoalgebra. Fortunately, the "natural" and the "correct" definitions almost coincide. If we use results like

LEMMA 2.8. <u>Suppose that</u> $B$ <u>is a closed subspace of the Banach space</u> $A$.

i) <u>If</u> $B$ <u>has the BAP, then</u> $(A \check{\otimes} B) \cap (B \check{\otimes} A) = A \check{\otimes} A$.

ii) <u>If</u> $A$ <u>has the BAP, then</u> $(A \check{\otimes} B) \cap (B \check{\otimes} A) = A \check{\otimes} A$ <u>iff</u> $B$ <u>has the bap</u>

it is not difficult to prove

PROPOSITION 2.9. <u>Suppose that</u> $J$ <u>is a closed subspace of the injective coalgebra</u> $C$.

(i) <u>If</u> $J$ <u>is a natural left coideal, then</u> $J$ <u>is a left coideal</u>.

<u>If either</u> $C$ <u>or</u> $J$ <u>has the BAP and if</u> $J$ <u>is a left coideal, then</u> $J$ <u>is a natural left coideal</u>.

(ii) <u>If</u> $J$ <u>is a natural subcoalgebra, the,</u> $J$ <u>is a subcoalgebra</u>. <u>If</u> $J$ <u>has the BAP (or if</u> $C$ <u>has the BAP and</u> $J$ <u>has the bap) and if</u> $J$ <u>is a subcoalgebra,</u>

then J is a natural subcoalgebra.

(iii) If J is a natural coideal, then J is a coideal. If C has the BAP, if J is a complemented subspace of C and if J is a coideal, then J is a natural coideal.

Before giving the proof of theorem 2.6, we isolate

LEMMA 2.10. An injective coalgebra C may be realised isometrically as a natural subcoalgebra of C(S), S some compact semigroup. S has an identity iff C is counital, and S is abelian iff C is cocommutative.

Proof. We take S to be the unit ball of C'. By 2.5, S is a compact semigroup. Define

$$\Phi : C \longrightarrow C(S) \quad ; \quad \Phi(c)(s) = \langle c,s\rangle \qquad \forall c \in C \quad \forall s \in S.$$

$\Phi$ is a linear isometry. It suffices to show that $\Phi$ is a coalgebra contraction. Writing $n$ for the comultiplication of $C(S)$, we have $(n\Phi(c))(s,t) = \Phi(c)(st) = \langle c,st\rangle = \langle Nc , s\otimes t\rangle \quad \forall c \in C \quad \forall s,t \in S$. The result follows immediately.

COROLLARY. The atoms of C are realised as semicharacters of S.

Proof of theorem 2.6.

(i) Keep the notations of 2.10, fix $c \in C$ and $s \in S$ and define $_s(\Phi(c)) \in C(S)$ by

$$_s(\Phi(c))(t) = \Phi(c)(st) \qquad \forall t \in S.$$

Then $_s(\Phi(c))(t) = \langle c , st\rangle = \langle s \rhd c , t\rangle = \Phi(s \rhd c)(t)$. Thus $_s(\Phi(c)) = \Phi(s \rhd c)$ and so $\Phi(C)$ is invariant under left translations. Right translation invariance is treated similarly.

(ii) Let $C$ be the space in question and take $f \in C$ and $\mu \in C^o \subseteq M(S)$. If $y \in S$, write $f_y$ for the right translate of $f$ by $y$. Then, by translation invariance, if $\nu \in M(S)$, we have



$$\langle Nf , \mu \otimes \nu \rangle = \int_S \int_S f(xy)\, d\mu(x)\, d\nu(y) = \int_S \langle f_y , \mu \rangle\, d\nu(y) = 0 .$$

Consequently $\langle f \lhd \nu , \mu \rangle = 0$ and so, by the bipolar theorem, $f \lhd \nu \in C$. Thus $C \lhd M(S) \subseteq C$, i. e. $C$ is a right coideal of $C(S)$. Similarly, it is a left coideal. Since $C(S)$ has the BAP, we may use 2.9 (ii) to get the desired result.

## 3. INJECTIVE BIALGEBRAS.

In this section we investigate what happens when we attempt to combine algebra and coalgebra structures on a Banach space. It is sensible to attempt this combination in a coherent way, and this leads us to abandon the full generality of Banach algebras. The appropriate restriction is given below.

DEFINITION. The Banach algebra $R$ is called an <u>injective algebra</u> if the multiplication is a linear contraction $M : R \check{\otimes} R \to R$.

Versions of injective algebras were studied by Varopoulos [8,9]. Kaijser [5] has proved the very useful

THEOREM. <u>A unital injective algebra</u> $R$ <u>is a uniform algebra, i. e. a closed subalgebra of</u> $C(X)$, $X$ <u>some compact Hausdorff space. In particular</u> $R$ <u>is commutative.</u>

One sees at once that if $R$ is an injective algebra, then so is $R \check{\otimes} R$. The multiplication is the composite

$$(R\check{\otimes} R)\check{\otimes}(R\check{\otimes} R) \xrightarrow[I\check{\otimes} T\check{\otimes} I]{} R\check{\otimes} R\check{\otimes} R\check{\otimes} R \xrightarrow[M\check{\otimes} M]{} R\check{\otimes} R.$$

If we interpret an identity for $R$ as a linear map $u : \mathbb{C} \to R$ of norm $1$, the following result is immediate.

LEMMA 3.1. *Suppose that* $H$ *is a Banach space which is an injective coalgebra with comultiplication* $N$ *(and coidentity* $e$*) and an injective algebra with multiplication* $M$ *(and identity* $u$*). Then* $M$ *(and* $u$*) are coalgebra contractions iff* $N$ *(and* $e$*) are algebra contractions, i. e. norm-decreasing multiplicative linear maps.*

This prompts

DEFINITION. A triple $(H,M,N)$ satisfying the equivalent conditions of lemma 3.1 is called an *injective bialgebra*. A quintuple $(H,M,N,u,e)$ satisfying these conditions is called a *unital counital injective bialgebra*.

As examples we have

(1) $C(S)$, $S$ a compact semigroup, under the usual comultiplication and pointwise multiplication.

(2) $C(G)$, $G$ a compact abelian group, under convolution multiplication and the usual comultiplication. Françoise Piquard has pointed out that convolution makes $C(G)$ an injective algebra.

(3) $\ell^1$ under pointwise multiplication and the usual comultiplication.

For a proof that $\ell^1$ is an injective algebra under pointwise multiplication, see [8].

We shall show that the maximal ideal space of a unital counital injective bialgebra is a compact semigroup. In fact, this is a special case of a more general result. First we generalise 2.2.

PROPOSITION 3.2. *Suppose that* $C$ *is an injective coalgebra and that* $R$ *is an injective algebra. Then we may define a multiplication,* $*$ *, on* $L(C,R)$ *under which it is a Banach algebra. If* $C$ *is cocommutative and* $R$ *is commutative, then* $L(C,R),*$ *is commutative. If* $C$ *is counital (with coidentity* $e$*) and* $R$ *is unital (with identity* $u$*), then* $L(C,R),*$ *is unital with identity* $u \circ e$.

*Proof*. If $\varphi, \psi \in L(C,R)$, it is easy to see that $\varphi * \psi$ may be appropriately defined as the composite

$$C \xrightarrow[N]{} C \check{\otimes} C \xrightarrow[\varphi \check{\otimes} \psi]{} R \check{\otimes} R \xrightarrow[M]{} R.$$

The example $C = C(S)$, $S$ a compact semigroup and $R = \mathbb{C}$ justifies the notation $*$.

Suppose now that $H$ is an injective bialgebra, that $C$ is a cocommutative injective coalgebra and that $R$ is a commutative injective algebra. Write $COALG_1(C,H)$ for the space of coalgebra contractions $C \to H$ and $ALG_1(H,R)$ for the algebra contractions $H \to R$. If we use the fact that the comultiplication $N : C \to C \check{\otimes} C$ is a coalgebra contraction (since $C$ is cocommutative) and that the multiplication $M : R \check{\otimes} R \to R$ is an algebra contraction (since $R$ is commutative), it is straightforward to verify

PROPOSITION 3.3. (i) $COALG_1(C,H)$ *is a closed topological subsemigroup of* $(L(C,H),*)$ *under the topology of pointwise convergence. It is abelian if* $H$ *is ; it is unital if* $H$ *is unital and* $C$ *is counital.*

(ii) $ALG_1(H,R)$ *is a closed topological subsemigroup of* $(L(H,R),*)$ *under* *the topology of pointwise convergence*. *It is abelian if* $H$ *is cocommutative ; it is unital if* $H$ *is counital and* $R$ *is unital*.

The special case $R = \mathbb{C}$ yields

COROLLARY. *If* $H$ *is a unital counital injective bialgebra, then its maximal ideal space* $\mathfrak{M}_H$ *is a compact unital semigroup for the weak* $*$ *topology*. $\mathfrak{M}_H$ *is abelian if* $H$ *is cocommutative*.

Now, we may specialise 2.6 to obtain

THEOREM 3.4. (i) *A unital counital injective bialgebra is a closed translation invariant subalgebra of* $C(S)$, $S$ *some compact unital semigroup*.

(ii) *If a closed translation invariant subalgebra of* $C(S)$ $S$ *a compact unital semigroup, has the bap, then it is a unital counital injective bialgebra under the multiplication and comultiplication induced from* $C(S)$.

*Proof*. Proceed as for 2.6, replacing the dual unit ball by the maximal ideal space wherever necessary.

## 4. INJECTIVE HOPF ALGEBRAS.

We show that the appropriate generalisation of the algebraically useful notion of a Hopf algebra gives nothing more than the spaces $C(G)$, $G$ a compact group.

DEFINITION. Let $H$ be a unital counital injective bialgebra. A linear contraction $S$ in $L(H,H)$ is said to be an *antipode* if it is inverse to the identity endomorphism, $I$, under $*$. (Note that the identity of $L(H,H)$ is $u \circ e$ and *not* the identity endomorphism). A unital counital injective bialgebra with antipode is called an injective Hopf algebra.

EXAMPLE 4.1. If $G$ is a compact group, then $C(G)$ under the usual operations is an injective Hopf algebra. The antipode is given by $S : C(G) \to C(G)$ ; $S\varphi(g) = \varphi(g^{-1}) \quad \forall \varphi \in C(G) \quad \forall g \in G$.

Proof. $S$ is well-defined since inversion is a continuous operation. $(I * S)(\varphi)(g) = (M(I \check{\otimes} S)N)(\varphi)(g) = \langle (I \check{\otimes} S) N\varphi , \delta_g \otimes \delta_g \rangle = \langle N\varphi, \delta_g \otimes \delta_{g^{-1}} \rangle =$ $\langle \varphi, \delta_g * \delta_{g^{-1}} \rangle = \langle \varphi, \delta_1 \rangle = \varphi(1) \quad \forall \varphi \in C(G) \quad \forall g \in G$. Here $\delta_g$ denotes Dirac measure at $G$, and $1$ is the identity of $G$.

The crucial thing about injective Hopf algebras is that the maximal ideal space is not merely a compact semigroup, but is in fact a compact group. This is a consequence of the (essentially algebraic) observation that the antipode is a sort of involution.

LEMMA 4.2. The antipode $S$ of an injective Hopf algebra $H$ is an algebra contraction.

Proof. We must show that $M \circ (S \check{\otimes} S) = S \circ M : H \check{\otimes} H \to H$ and that $S \circ u = u : \mathbb{C} \longrightarrow H$.

The second equality is easy. The first follows from the fact that both mappings are inverse to $M$ in $L(H \check{\otimes} H , H), *$.

COROLLARY. $S \circ S = I$, i.e. $S$ be haves like an involution.

Proof. By the definition of $S$, it is enough to show that

$S * (S \circ S) = u \circ e$ in $L(H,H),*$. But $S * (S \circ S) = M(S \check{\otimes} S)(I \check{\otimes} S)N = SM(I \check{\otimes} S)N$ by 4.2. The result now follows from the definition of $S$.

We may now improve on 3.3 for the special case of injective Hopf algebras.

PROPOSITION 4.3. Let $H$ be an injective Hopf algebra and let $R$ be a unital injective algebra. Then $ALG_1(H,R)$ is a topological group under the multiplication $*$ of $L(H,R)$.

Proof. By 3.3, we need only produce inverses. We show that if $S$ is the antipode of $H$, then the inverse of $\Psi \in ALG_1(H,R)$ is $\Psi \circ S$.

If $f \in L(H,H)$, write $\psi(f) = \Psi \circ f \in L(H,R)$. Then $\psi : L(H,H) \to L(H,R)$ is an algebra contraction (by the definition of $\Psi$) for the multiplications $*$. Now we know that $S = I^{-1}$ in $L(H,H),*$. Thus $\psi(S) = \psi(I)^{-1}$, i. e. $\Psi \circ S = \Psi^{-1}$. The continuity of the inverse operation is clear.

COROLLARY. The maximal ideal space of a (cocommutative) injective Hopf algebra is a compact (abelian) group in the weak $*$ topology.

Finally we arrive at the main result of this section.

THEOREM 4.4. Every injective Hopf algebra $H$ may be identified (as an injective Hopf algebra) with the space of continuous functions on its maximal ideal space $G$.

Proof. We suppose for simplicity that $G$ is abelian, and write $\hat{G}$ for the character group of $G$. To treat the non abelian case, one has simply to translate the proof below into the language of the representation theory for compact groups [4].

As we have seen in 3.4, H may be considered as a closed translation invariant subalgebra of C(G). Consequently, it has the form

$$C_X(G) = \left\{ f \in C(G) : \hat{f} \text{ is supported on } X \subseteq \hat{G} \right\}.$$

We shall use the Stone-Weierstrass theorem to show that in fact $H = C(G)$. Firstly, H clearly contains the constant functions. Secondly, the fact that G is the maximal ideal space of H implies at once that the functions in H separate the points of G. It remains to show that H is closed under complex conjugation.

Using 4.3, one may quickly verify that the antipode of H coïncides with the restriction to H of the antipode of C(G). Now $\chi \in X \subset H \Rightarrow S\chi \in H$. But $S\chi(g) = \chi(g^{-1}) = \overline{\chi(g)} \quad \forall g \in G$. Since every function in $C_X(G)$ may be uniformly approximated by finite linear combinations of elements of X, we have proved that $H = C_X(G)$ is closed under complex conjugation.

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