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P R E F A C E

The aim of this work is to present recent developments in the theory of the Riemann zeta-function. Apart from the renowned classic of E.C. Titchmarsh [8], there seems to exist only the book of H.M. Edwards [1] devoted solely to the topic of the Riemann zeta-function. Despite the undisputable merits of these works, many recent developments in zeta-function theory make it desirable to have a text which contains a systematic account of recent developments in the theory. The general high quality of Titchmarsh's book makes it unnecessary to repeat most of the material presented there, and my purpose in writing this text was more to continue where Titchmarsh's work stopped, than to provide a complete and systematic account of the whole theory of the Riemann zeta-function. However efforts have been made to make the text as self-contained as possible while keeping its length moderate, so that it does not seem absolutely necessary for the reader to know Titchmarsh's book in detail, although a standard knowledge of complex analysis and basic zeta-function theory is required. In this way it seems that the text will be of interest also to those who are not experts in the field, but do wish to get acquainted with recent developments of the subject.

The great abundance and depth of the existing material naturally set a limitation to the size and scope of this text; hence the title "Topics in recent zeta-function theory", since the work does not pretend to cover all important aspects of modern zeta-function theory. A word will be said now about what is and what is not included in this text. As in Titchmarsh's book [8], no prime number theory is touched, although this topic is intimately connected with the zeta-function. The material concerning prime numbers (see for instance the standard work of K. Prachar [1], which does not contain many new results which appeared subsequently) is so vast that it certainly cannot be adequately covered today within a single volume together with the zeta-function, and one certainly feels that it ought to be treated separately. Another important topic closely related to the zeta-function which is also omitted both by Titchmarsh [8] and here is the theory of L-functions and related more general Dirichlet series. A great richness of material also exists here, but there is another important reason which makes the Riemann zeta-function worth being treated separately and gives it a unique position among all L-functions. Namely the absence of the analogue of Atkinson's formula (see Chapter 11) for $\int_0^T |\zeta(1/2+it)|^2 dt$ for L-functions makes a number of very important new results impossible to obtain at present for L-functions. It has become fashionable during the last 10-15 years to treat the Riemann zeta-function and L-functions often together (especially in zero-density estimates), but recent results on $\zeta(s)$ make it doubtful whether such a unified approach is worthwhile.

A word must be said now about two very famous classical conjectures of zeta-function theory which are also not discussed systematically in this text. As the reader has probably guessed, the conjectures in question are Lindelöf's ($\zeta(1/2+it) \ll |t|^{\varepsilon}$) and Riemann's (all non-trivial zeros of the zeta-function have real parts equal to 1/2). An extensive discussion of these conjectures and their consequences has been given in Chapters 13 and 14 of E.C. Titchmarsh [8], which represent one of the high points of his book. As is well-known, both of these conjectures (Riemann's implies Lindelöf's) are even today neither proved nor disproved. Despite some important new results (like N. Levinson's paper [1] that more than a third of zeros of $\zeta(s)$ lie on the line $\mathrm{Re}\, s = 1/2$) and impressive numerical evidence (R. Brent [1] stated that the first 75 000 001 zeros of $\zeta(s)$ are simple and on the critical line), the Riemann hypothesis is in some ways as remote as ever - witness the fact that one cannot prove yet the estimate $\int_0^T |\zeta(1/2+it)|^k dt \ll T^{1+\varepsilon}$ for any $k > 4$, and this estimate would follow already from the Lindelöf hypothesis. Another interesting conjecture has been made fairly recently by H.L. Montgomery [3] (see D.A. Goldston [1] for some applications), and bears the name "the pair correlation conjecture". However I have found it preferable to deal in general only with unconditional results, leaving aside conjectures like Lindelöf's or Riemann's of which personally I disbelieve the latter one.

The choice of topics which are covered in a work such as this one must be highly selective, and at the end in some sense personal. It is to be regretted that all important recent results in zeta-function theory could not find their place here: some because of the length of the proofs (like the aforementioned result of N. Levinson [1]) and some for various other reasons. For example I have felt that the proof of the best known zero-free region (see A. Walfisz [3]), namely $\sigma \geq 1 - C\log^{-2/3}|t|(\log\log|t|)^{-1/3}$, does not necessarily deserve to be given here. The result is not so new really, and besides its proof comes only from a more careful application of I.M. Vinogradov's classical method of the estimation of exponential sums than was done earlier. Some important results not fully treated in the text are mentioned with due references in Notes at the end of each chapter (except the first, which is of an introductory nature). These Notes also contain historical discussion, elucidation of certain details in proofs, etc.

After this apologyabout the topics that have been omitted it seems appropriate to discuss briefly the material that has been given in the text. It might be said that the general systematic approach is of the "Voronoï-Atkinson" type, since it turns out that the formulas of Voronoï (Chapter 3) and Atkinson (Chapter 11) play a prominent role in recent zeta-function theory. Problems are often reduced to an estimation of a finite exponential sum which often may be treated either directly by (variants of) van der Corput's method or by Voronoï's summation formula, and the quality of the final result depends on our capability to estimate the exponential sums in question. Pure zeta-function theory begins with Chapter 4, which presents various approximate functional equations. The first chapter contains loosely connected analytic results and formulas which are often used in the sequel, while Chapter 2 contains a rather extensive treatment of exponential sums and integrals. Van der Corput's theory of exponent pairs (in a simplified form) is fully explained, and is used later on several occasions.

Chapter 3 is devoted to Voronoï's summation formula and related problems. This topic requires the knowledge of the theory of Bessel functions, and all the facts about Bessel functions that are used in this text may be found in G.N. Watson's standard treatise [1] on the subject.

Various approximate functional equations are discussed in Chapter 4, which contains the author's hitherto unpublished material on the approximate functional equation for $\zeta^k(s)$. Also given there is the approximate functional equation based on the so-called "reflection principle", which is very useful in zero-density estimates.

Chapter 5 presents a proof of the fourth power moment for the zeta-function. Although the sharpest known result due to D.R. Heath-Brown [3] is not given, the difficult classical asymptotic formula of A.E. Ingham [1] is proved in a relatively simple way, using an approach due to K. Ramachandra [3], [5].

The estimates for $\int_{T-G}^{T+G} |\zeta(\sigma+it)|^2 dt$, $G = o(T)$ are discussed in Chapter 6, and these estimates form the basis of many results of later chapters. The proof of the important Theorem 6.2, originally given by Heath-Brown [1] with the aid of Atkinson's formula, is based here on the use of Voronoï's formula. Order estimates for $\zeta(s)$ in the critical strip are also given, including G. Kolesnik's estimate [6] that $\zeta(1/2 + it) \ll t^{35/216+\varepsilon}$.

One of the parts of this work which shows best how much zeta-function theory has advanced since the days of Titchmarsh's classic [8] is Chapter 7, which deals with estimates for power moments of the zeta-function higher than the fourth. Based mainly on author's paper [2], this chapter gives among other things the important estimate $\int_0^T |\zeta(1/2+it)|^{12} dt \ll T^2 \log^{17} T$ of D.R. Heath-Brown [1]. In spite of many recent results in this area of research one feels that still much more can be done in the future.

Chapter 8 is concerned with estimates of $\gamma_{n+1} - \gamma_n$, the difference of ordinates of consecutive zeros of the zeta-function on the critical line. The author's result $\gamma_{n+1} - \gamma_n \ll \gamma_n^{u+\varepsilon}$, $u = 0.1559458...$ is presented as the limit of a certain method based on the use of the theory of exponent pairs. This chapter is virtually independent of other chapters, except Chapter 2.

Zero-density estimates are treated in Chapter 9. All the best-known results for $N(\sigma,T)$ (except when σ is very close to 1/2 or to 1) are given, and the importance of power moments for the estimation of $N(\sigma,T)$ is stressed. The modern flexible methods which use the Halász-Montgomery inequality have proved very effective in the range $\sigma > 3/4$, and variants of this approach are extensively discussed.

Chapter 10 is the analogue of Chapter 12 of Titchmarsh [8], and is devoted to divisor problems. I have shared Titchmarsh's viewpoint that divisor problems should be included in a work on the zeta-function of Riemann, and recent investigations of M. Jutila [4], [5], [6] certainly confirm the validity of this viewpoint. It is now evident that a direct connection between $\Delta(x)$ and $\zeta(1/2 + it)$ exists, an analogy also suggested by Atkinson's formula. Thus it turns out that divisor problems are an important and intrinsic part of zeta-function theory. Besides achieving an almost overall improvement of results given in Chapter 12 of Titchmarsh [8] (some hitherto unpublished material is included) the chapter contains a discussion of the circle problem. I have tried to give a unified approach to the three classical problems of analytic number theory, namely the circle problem, the ordinary divisor problem, and the problem of the order of $\zeta(1/2 + it)$. The existence of a truncated Voronoï-type formula in all three of these problems (for the zeta-function this is Theorem 6.2 really) makes all three problems very similar as they can be reduced to the estimation of analogous exponential sums. Further evidence for this viewpoint is contained in Chapter 11, where power moments for $E(x)$ are derived, which are the exact analogues of the corresponding estimates for $\Delta(x)$ and $P(x)$ in Chapter 10.

The last chapter is Chapter 11, which is centered around the single deepest result of this text - Atkinson's formula for $E(T)$ - and some of its applications. It is to be regretted that this beautiful formula has been neglected for a very long time, until Heath-Brown [2] made the first important application to the mean square of $E(t)$ and in [1] to the twelfth power moment estimate. Several applications of Atkinson's formula are considered in Chapter 11, and it is certain that the possibilities of Atkinson's formula are far from being exhausted.

Finally I wish to thank all number-theorists who have read (parts of) the manuscript and made valuable remarks, especially D.R. Heath-Brown, M.N. Huxley, M. Jutila and H.-E. Richert.

Belgrade, November 1982.

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NOTATION

Owing to the nature of this text no absolute consistency in notation could have been attained. Notation commonly used throughout the text is explained here, while specific notation introduced in the proof of a theorem or lemma is given at the proper place in the body of the text.

l, k, m, n : natural numbers (positive integers).

s, z, w : complex variables ($\mathrm{Re}\, s$ and $\mathrm{Im}\, s$ denote the real and imaginary part of s respectively; common notation $\sigma = \mathrm{Re}\, s$ and $t = \mathrm{Im}\, s$).

$\mathrm{Res}_{s=s_0} F(s)$: denotes the residue of the function $F(s)$ at the point $s = s_0$.

$\zeta(s)$: Riemann's zeta-function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}\, s > 1$ and otherwise by analytic continuation.

$\Gamma(z)$: the gamma-function is defined by $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$ for $\mathrm{Re}\, z > 0$, otherwise by analytic continuation.

$\exp z$: $= e^z$.

$e(z)$: $= e^{2\pi i z}$.

t, x, y : real variables.

γ : Euler's constant, defined by $\gamma = -\int_0^{\infty} e^{-x} \log x \cdot dx = 0.5772157\ldots$.

$\chi(s)$: the function defined by $\zeta(s) = \chi(s)\zeta(1-s)$, so that by the functional equation for the zeta-function $\chi(s) = (2\pi)^s/(2\Gamma(s)\cos(\pi s/2))$.

$A, C, C_1, \ldots$: absolute positive constants (not necessarily the same at each occurrence in a proof).

$[x]$: the greatest integer not exceeding the real number x.

$\sum_{n\le x} f(n)$: a sum taken over all natural numbers not exceeding x; the empty sum is defined to be zero.

$\sum_{n\le x}{}' f(n)$: the same as above, only ' denotes that when x is an integer one should take the last term in the sum as $\frac{1}{2} f(x)$ and not as $f(x)$.

$d_k(n)$: the number of ways n can be written as a product of $k \ge 2$ fixed factors; $d_2(n) = d(n)$ is the number of divisors of n.

$r(n)$: the number of ways n can be written as a sum of two integer squares.

$\mu(n)$: the Möbius function, defined by $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$ ($\mathrm{Re}\, s > 1$).

$\Delta_k(x)$: $= \sum_{n\le x} d_k(n) - \operatorname{Res}_{s=1} x^s \zeta^k(s)s^{-1}$ for $k \ge 3$, $\Delta_2(x) = \Delta(x)$ in Chapter 10.

$\Delta(x)$: $= \sum_{n\le x}{}' d(n) - x\log x - (2\gamma-1) - 1/4$ (but see Chapter 10 for a modified definition).

$P(x)$: $= \sum_{n\le x}{}' r(n) - \pi x - 1$ (but see Chapter 10 for a modified definition).

$\Lambda(n)$: the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n = p^m$ (p prime) and zero otherwise.

$J_p(z), K_p(z), Y_p(z)$: notation for the Bessel functions of index p defined in Chapter 3.

$E(T)$: $= \int_0^T |\zeta(1/2 + it)|^2 dt - T\log(T/2\pi) - T(2\gamma-1)$.

$N(\sigma,T)$: denotes the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ (β,γ real) for which $\beta \ge \sigma \ge 0$, $-T \le \gamma \le T$.

ar sinh z : $= \log(z + \sqrt{z^2 + 1})$.

(p,q) : an exponent pair (a certain pair of real numbers for which $0 \le p \le 1/2 \le q \le 1$; precise definition and properties are given in Chapter 2).

$\psi(x)$: $x - [x] - 1/2$ (but only in Chapter 3 $\psi(z) = \Gamma'(z)/\Gamma(z)$, while in Notes of Chapter 9 $\psi(x) = \sum_{n\le x}\Lambda(n)$).

$f(x) \sim g(x)$ as $x \to x_0$: means $\lim_{x\to x_0} f(x)/g(x) = 1$.

$f(x) = O(g(x))$: means $|f(x)| \le Cg(x)$ for $x \ge x_0$ and some absolute constant $C > 0$. Here $f(x)$ is a complex function of a real variable and $g(x)$ is a positive function for $x \ge x_0$.

$f(x) \ll g(x)$: means the same as $f(x) = O(g(x))$.

$f(x) \asymp g(x)$: means that both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

(a,b) : means the interval $a < x < b$.

$[a,b]$: means the interval $a \le x \le b$.

δ, ε : an arbitrarily small positive number, not necessarily the same at each occurrence in the proof of a theorem or lemma.

$C^r[a,b]$: **the class of functions having a continuous r-th derivative in** $[a,b]$.

$f(x) = o(g(x))$: means that for each $\varepsilon > 0$ there exists x_0 such that $|f(x)| < \varepsilon g(x)$ for $x \geq x_0$, where $g(x)$ is a positive function for $x \geq x_0$.

$f(x) = \Omega_+(g(x))$: means that there exists a suitable constant $C > 0$ such that $f(x) > Cg(x)$ holds for a sequence $x = x_n$ with $\lim_{n\to\infty} x_n = \infty$.

$f(x) = \Omega_-(g(x))$: means that there exists a suitable constant $C > 0$ such that $f(x) < -Cg(x)$ holds for a sequence $x = x_n$ with $\lim_{n\to\infty} x_n = \infty$.

$f(x) = \Omega_\pm(g(x))$: means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ holds.

$f(x) = \Omega(g(x))$: means that $|f(x)| = \Omega_+(g(x))$.

$c(\theta)$: for real θ defined by $\zeta(\theta + iT) \ll T^{c(\theta)+\varepsilon}$ for any $\varepsilon > 0$ and $T \geq T_0(\varepsilon)$.

Aleksandar Ivić

TOPICS IN RECENT ZETA FUNCTION-THEORY

C H A P T E R 1

I N T R O D U C T O R Y R E S U L T S

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CHAPTER 1

INTRODUCTORY RESULTS

§1. Introduction

This chapter contains preliminary results which will be repeatedly used in later chapters. Most of the material consists of well-known analytic facts which are given here as a reference for the sake of completeness of the exposition. This seems preferable to quoting these results from the literature each time the need for such a result arises at a specific place in the body of the text. The material presented in this chapter is only loosely connected, and the choice for its inclusion is solely motivated by needs of later chapters. For this reason detailed proofs are not given, and sometimes only a reference to a standard text is offered, where proofs and a more detailed account may be found. It is clear that the criteria for deciding what is well-known and what is not are highly personal, so that it may occur to the reader that some additional material should have been included here, while some could have been omitted.

As in the whole text, standard notation is being used, albeit absolute consistency in notation can hardly be ever achieved. Whenever possible the notation of E.C. Titchmarsh's book [8] on the zeta-function is used, and although it is not absolutely necessary it will help the reader if he is familiar with the contents of Titchmarsh's book. One of the results which certainly belongs here in Chapter 1 is Voronoï's summation formula, but due to its complexity and importance this formula will be treated separately in Chapter 3. As Chapter 2 is devoted to exponential sums and integrals, it may be justly said that pure zeta-function theory begins with Chapter 4.

§2. Mellin transforms

Let $f(x)x^{\sigma-1}$ belong to $L(0,\infty)$ and let $f(x)$ have bounded variation in every finite x-interval. Then

$$F(s) = \int_0^\infty x^{s-1} f(x)\,dx, \quad s = \sigma + it \quad (\sigma, t \ \text{real}) \tag{1.1}$$

is defined as the Mellin transform of $f(x)$. From (1.1) we can recover $f(x)$ by Mellin's inversion formula

(1.2) $$\frac{1}{2}(f(x+0) + f(x-0)) = (2\pi i)^{-1} \lim_{T\to\infty} \int_{\delta-iT}^{\delta+iT} F(s)x^{-s}ds.$$

In the case when f(x) is continuous (1.2) can be obtained without difficulty from (1.1), while in the general case it seems more suitable to write (1.1) as a Fourier transform by a change of variable and then to appeal to results from the theory of Fourier transforms and integrals. A detailed account of (1.1) and (1.2) is to be found in E.C. Titchmarsh's book [7] on Fourier integrals. Relations (1.1) and (1.2) are inverse to one another. Namely if

(1.3) $$f(x) = (2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} F(s)x^{-s}ds,$$

where $F(\delta + iu)$ belongs to $L(-\infty, \infty)$ and is of bounded variation in the neighborhood of the point $u = t$ and (1.3) holds, then

(1.4) $$\frac{1}{2}\{F(\delta + i(t+0)) + F(\delta + i(t-0))\} = \lim_{a\to\infty} \int_{1/a}^{a} f(x)x^{\delta+it-1}dx,$$

and in most applications (1.4) will reduce to (1.1).

An analogue of the well-known Parseval's identity for Fourier integrals holds also for Mellin transforms; i.e. if f and F are connected by (1.1), then

(1.5) $$(2\pi)^{-1} \int_{-\infty}^{\infty} |F(\delta + it)|^2 dt = \int_0^{\infty} f^2(x)x^{2\delta-1}dx.$$

As is the case with (1.2), this identity may be derived from Parseval's identity for Fourier transforms, or one may argue directly by writing

$$(2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} F(s)\overline{F(s)}ds = \int_0^{\infty} f(x)((2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} F(s)x^{\delta-it-1}ds)dx =$$

$$\int_0^{\infty} f(x)x^{2\delta-1}((2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} F(s)x^{-s}ds)dx = \int_0^{\infty} f^2(x)x^{2\delta-1}dx,$$

where (1.1) and (1.2) were used under the assumption that f is continuous. Setting $s = \delta + it$ we obtain (1.5).

Formulas analogous to (1.5) hold also for two or more functions. As an example, suppose that F(s) and G(s) are Mellin transforms of two continuous functions f(x) and g(x) respectively. Then

(1.6) $$(2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} F(s)G(1-s)ds = (2\pi i)^{-1} \int_{\delta-i\infty} G(1-s)(\int_0 f(x)x^{s-1}dx)ds =$$

$$(2\pi i)^{-1} \int_0^\infty f(x)dx \int_{\delta-i\infty}^{\delta+i\infty} G(1-s)x^{s-1}ds = \int_0^\infty f(x)g(x)dx.$$

Finally it may be mentioned that inversion of the gamma-integral (see §6)

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx, \quad (\mathrm{Re}\, s > 0)$$

gives by (1.2) the useful relation

$$(1.7) \qquad e^{-x} = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds. \qquad (c, x > 0)$$

§3. Inversion formulas for Dirichlet series

We shall consider Dirichlet series of the form $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ which have a finite abscissa of absolute convergence, and we shall set $f(x) = \sum_{n\le x}' a_n$. General theory of Dirichlet series will not be discussed here, since our main interest lies in inversion formulas, which represent formulas expressing $f(x)$ (or some similar function involving the a_n's) by series and integrals containing $A(s)$. Sometimes these formulas go under the name of "Perron's formula", although this name is most often used for one particular formula of this sort, namely

$$(1.8) \qquad \sum_{n\le x}{}' a_n = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} A(s)x^s s^{-1}ds,$$

where $c > 0$ is such a number that $A(s)$ is absolutely convergent for $\mathrm{Re}\, s = c$. Here $\sum'$ means that if x is an integer then $\frac{1}{2}a_n$ comes instead of a_n into the sum. One obtains (1.8) easily from

$$(1.9) \qquad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} Y^s s^{-1}ds = \begin{cases} 0 & 0 \le Y < 1 \\ 1/2 & Y = 1 \\ 1 & Y > 1 \end{cases}, \qquad (c > 0)$$

since in view of absolute convergence of $A(s)$ one may integrate term by term the right-hand side of (1.8) to obtain using (1.9)

$$(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} A(s)x^s s^{-1}ds = \sum_{n=1}^{\infty} a_n (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} (x/n)^s s^{-1}ds = \sum_{n\le x}{}' a_n.$$

To see that (1.9) holds one may evaluate the integral directly by the residue theorem, or defining $f(x) = 0$ for $0 \le x < 1$, $f(1) = 1/2$, $f(x) = 1$ for $x > 1$ one has the Mellin transform

$$F(s) = \int_1^{\infty} f(x)x^{s-1}dx = x^s s^{-1}\Big|_1^{\infty} = -s^{-1} \qquad (\mathrm{Re}\, s < 0)$$

and (1.9) follows from (1.2) on replacing x by Y and s by $-s$. Instead of (1.8) it is often desirable to have a truncated form of the inversion formula, namely a formula where the integral is over a finite segment whose length may be suitably chosen. Such a formula may be obtained by considering the integral in (1.8) and replacing it by an integral over a suitable finite contour plus an error term. Various hypotheses on $A(s)$ make then a more explicit evaluation of $\sum'_{n\le x} a_n$ possible.

We quote now a standard result of this type, whose details of proof may be found in K. Prachar [1] :

Let $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converge absolutely for $\sigma = \mathrm{Re}\, s > 1$ and let $|a_n| < C\phi(n)$, where for x large $\phi(x)$ is a monotonically increasing function. Let further

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \ll (\sigma - 1)^{-\alpha}$$

as $\sigma \to 1+0$ for some $\alpha > 0$. If $w = u + iv$ (u,v real) is arbitrary, $b,T > 0$, $u + b > 1$, then

$$(1.10) \qquad \sum_{n\le x}{}' a_n n^{-w} = (2\pi i)^{-1} \int_{b-iT}^{b+iT} A(s + w)x^s s^{-1}ds + O(x^b T^{-1}(u+b-1)^{-\alpha}) +$$

$$+ O(T^{-1}\phi(2x)x^{1-u}\log 2x) + O(\phi(2x)x^{-u}),$$

and the estimate is uniform in x,T,b,u provided that b and u are bounded.

Another inversion formula for Dirichlet series is

$$(1.11) \qquad \sum_{n\le x} a_n n^{-w}\log^{k-1}(x/n) = (2\pi i)^{-1}(k-1)! \int_{c-i\infty}^{c+i\infty} A(s + w)x^s s^{-k}ds, \quad (c > 0)$$

where $k \ge 2$ is a fixed integer, w is an arbitrary complex number and $c + \mathrm{Re}\, w$ exceeds the abscissa of absolute convergence of $A(s)$. This formula is obtained if one integrates term by term the right-hand side of (1.11) with the use of

$$(1.12) \qquad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} Y^s s^{-k}ds = \begin{cases} 0, & 0 \le Y < 1 \\ \frac{1}{(k-1)!}\log^{k-1} Y, & Y > 1 \end{cases} \qquad (c > 0)$$

where $k \ge 2$ is a fixed integer. To see that (1.12) holds one may start from

$$\int_0^\infty e^{-sx}x^{k-1}dx = s^{-k}(k-1)!, \quad \mathrm{Re}\, s > 0$$

and make the change of variable $e^{-x} = u$ to obtain

$$\frac{1}{(k-1)!}\int_0^1 u^{s-1}(-1)^{k-1}\log^{k-1}u\cdot du = s^{-k}.$$

The inversion formula (1.2) gives then

$$(1.13) \qquad (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} u^{-s}s^{-k}ds = \begin{cases} \frac{(-1)^{k-1}}{(k-1)!}\log^{k-1}u, & 0 < u < 1 \\ 0, & u \geq 1 \end{cases}$$

so that (1.12) follows with $Y = u^{-1}$.

Finally we present an inversion formula for a weighted sum which differs from the one appearing in (1.11). We suppose that $q > 0$ is a fixed real number and that $A(s)$ converges absolutely for $\mathrm{Re}\, s = c > 0$. Then

$$(1.14) \qquad \frac{1}{\Gamma(q+1)}\sum_{n\leq x} a_n(x-n)^q = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)A(s)}{\Gamma(s+q+1)}x^{s+q}ds,$$

and in case when $q = 0$ (1.14) reduces to (1.8) in view of $\Gamma(s+1) = s\Gamma(s)$. One may obtain (1.14) by termwise integration of the right-hand side with the aid of

$$(1.15) \qquad (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \frac{u^{-s}\Gamma(s)}{\Gamma(s+q+1)}ds = f(u) = \begin{cases} \frac{(1-u)^q}{\Gamma(q+1)}, & 0 < u \leq 1 \\ 0, & u > 1 \end{cases} \qquad (c > 0)$$

when one replaces u by n/x. To see that (1.15) holds start from (1.29), namely

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b+1)} \qquad (\mathrm{Re}\, a > 0, \mathrm{Re}\, b > 0)$$

to obtain

$$(1.16) \qquad F(s) = \int_0^\infty f(x)x^{s-1}dx = \int_0^1 \frac{(1-x)^q x^{s-1}}{\Gamma(q+1)}dx = \frac{\Gamma(q+1)\Gamma(s)}{\Gamma(q+1)\Gamma(q+s+1)},$$

which shows that $\Gamma(s)/\Gamma(q+s+1)$ is the Mellin transform of $f(x)$, and consequently (1.15) follows from the inversion formula (1.2).

§4. Partial summation formulas

Partial summation is a standard elementary technique for transforming sums into more manageable sums or integrals, and some of these useful formulas are recorded here as a reference.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and $\{b_n\}_{n=1}^{\infty}$ a sequence of real numbers. If $b_1 \geq b_2 \geq \ldots \geq 0$ and $M \geq 1$ is an integer, then

$$(1.17) \qquad \Big|\sum_{M<n\leq N} a_n b_n\Big| \leq b_M \max_{M<n\leq N} \Big|\sum_{M<m\leq n} a_m\Big|,$$

while if $0 \leq b_1 \leq b_2 \leq \ldots$, then

$$(1.18) \qquad \Big|\sum_{M<n\leq N} a_n b_n\Big| \leq 2b_N \max_{M<n\leq N} \Big|\sum_{M<m\leq n} a_m\Big|.$$

These simple inequalities show that monotonic sequences may be removed from sums, and they are both proved analogously. To obtain (1.17) we define $A_n = \sum_{M<m\leq n} a_m$. Then

$$\Big|\sum_{M<n\leq N} a_n b_n\Big| = \Big|\sum_{M<n\leq N} (A_n - A_{n-1}) b_n\Big| \leq$$

$$|A_N| b_N + \sum_{M<n\leq N-1} |A_n| (b_n - b_{n+1}) \leq b_M \max_{M<n\leq N} |A_N|.$$

To transform sums into integrals it is often convenient to write a sum as a Stieltjes integral and then to integrate by parts. For example if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, $g(x) \in C^1[\lambda_1, x]$, $\lambda_1 \leq \lambda_2 \leq \ldots$ is a sequence of real numbers tending to infinity, then

$$(1.19) \qquad \sum_{\lambda_1 \leq \lambda_n \leq x} a_n g(\lambda_n) = A(x)g(x) - \int_{\lambda_1}^{x} A(t)g'(t)\,dt,$$

where

$$(1.20) \qquad A(t) = \sum_{\lambda_1 \leq \lambda_n \leq t} a_n.$$

Namely we can write

$$\sum_{\lambda_1 \leq \lambda_n \leq x} a_n g(\lambda_n) = \int_{\lambda_1 - 0}^{x+0} g(t)\,dA(t),$$

since $A(t)$ has jumps of weight a_n for $t = \lambda_n$ and otherwise it is a constant function. An integration by parts yields immedialtely (1.19), since

$$g(t)A(t)\Big|_{\lambda_1-0}^{x+0} = g(x)A(x) - g(\lambda_1-0)A(\lambda_1-0) = g(x)A(x).$$

Similarly we can obtain

(1.21) $$\sum_{X<n\le Y} f(n) = \int_X^Y f(t)dt - \psi(Y)f(Y) + \psi(X)f(X) + \int_X^Y \psi(t)f'(t)dt,$$

where $f(x) \in C^1[X,Y]$ and $\psi(t) = t - [t] - 1/2$. This is a special case of the so-called Euler-Maclaurin summation formula, and essentially only a variant of (1.19). The general Euler-Maclaurin formula is (for simplicity we shall assume here that a and b are integers)

(1.22) $$\sum_{a<k<b} f(k) = \int_a^b f(t)dt + \frac{1}{2}(f(a)+f(b)) + \sum_{m=1}^{n}\frac{B_{2m}}{(2m)!}(f^{(2m-1)}(b)-f^{(2m-1)}(a))$$

$$+ \int_a^b P_{2n+1}(t)f^{(2n+1)}(t)dt, \qquad (n \ge 0)$$

Here $f(x) \in C^{2n+1}[a,b]$, B_m is the m-th Bernoulli number and P_m is the m-th periodic Bernoulli function defined by $P_m(x) = B_m(x - [x])$, where $B_m(x)$ is the Bernoulli polynomial defined by

$$\frac{ze^{xz}}{e^z-1} = \sum_{m=0}^{\infty} B_m(x)z^m/m!, \qquad (|z| < 2\pi)$$

so that $B_m = B_m(0)$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, etc. A proof of (1.22) may be obtained as follows: by the Stieltjes integral representation and integration by parts we have

$$\sum_{a\le k\le b} f(k) = \int_{a-0}^{b+0} f(t)d([t]) = \int_a^b f(t)dt + \int_{a-0}^{b+0} f(t)d([t]-t+1/2) =$$

$$\int_a^b f(t)dt - \int_{a-0}^{b+0} f(t)dP_1(t) = \int_a^b f(t)dt + \frac{1}{2}(f(a)+f(b)) + \int_a^b P_1(t)f'(t)dt,$$

which is (1.21) for $a = [X]+1, b = [Y]$. From the defining property of Bernoulli polynomials (see T.M. Apostol [1], Ch. 12) $B'_{n+1}(x) = (n+1)B_n(x)$, so that one may take $\int P_n(x)dx = (n+1)^{-1}P_{n+1}(x)$, and repeated integration by parts of $\int_a^b P_1(t)f'(t)dt$ leads to (1.22), since for any integer r we have $P_m(r) = B_m(0) = B_m$ and $B_{2m+1} = 0$ for $m \ge 1$.

§5. The Poisson summation formula

There exist several variants of this useful formula. We shall need the following version: let a,b be integers and let f(x) be a function of a real variable with bounded first derivative on $[a,b]$. Then

$$(1.23)\qquad \sum_{a\le n\le b}{}' f(n) = \int_a^b f(x)\,dx + 2\sum_{n=1}^{\infty}\int_a^b f(x)\cos 2\pi nx\cdot dx.$$

Here as usual $\sum'$ means that $\frac{1}{2}f(a)$ and $\frac{1}{2}f(b)$ are to be taken instead of f(a) and f(b) respectively. To derive (1.23) we use (1.21) in the form

$$\sum_{a\le n\le b} f(n) = \int_a^b f(x)\,dx + \int_a^b \psi(x)f'(x)\,dx,$$

and thus we have to show that

$$(1.24)\qquad \int_a^b \psi(x)f'(x)\,dx = 2\sum_{n=1}^{\infty}\int_a^b f(x)\cos 2\pi nx\cdot dx.$$

This is achieved by using the Fourier series expansion

$$(1.25)\qquad \psi(x) = -\pi^{-1}\sum_{n=1}^{\infty} n^{-1}\sin 2\pi nx,$$

which is valid if x is not an integer. The series in (1.25) is equal to zero if x is an integer, and moreover by partial summation it is seen that its partial sums are uniformly bounded for any real x. Therefore using (1.25) in (1.24) and integrating by parts we obtain the right-hand side of (1.24) since $\sin 2\pi ma = \sin 2\pi mb = 0$.

A more detailed account of Poisson's summation formula may be found for instance in Chapter 10 of M.N. Huxley's book [1], where a good bound for the tails of the series in (1.23) is given.

§6. The gamma-function

Several standard properties of the gamma-function will be stated now (some were already used in §3). Their proofs may be found in standard books in analysis, and therefore no particular references will be given.

For $\mathrm{Re}\, s > 0$ the gamma-function is defined as

$$(1.26)\qquad \Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}\,dx,$$

and for other values of s by analytic continuation. $\Gamma(s)$ is an analytic function of s in the whole plane, except for points $s = 0,-1,-2,\ldots,-n,\ldots$ which are poles of the first order with residues $(-1)^n/n!$ $(n = 0,1,2,\ldots)$. The gamma-function satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s) \tag{1.27}$$

and the useful relations

$$\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s, \quad \Gamma(s)\Gamma(s+1/2) = 2\sqrt{\pi}\,2^{-2s}\Gamma(2s). \tag{1.28}$$

Another common property is

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \qquad (\mathrm{Re}\ a > 0, \mathrm{Re}\ b > 0) \tag{1.29}$$

We shall also make use of the relation

$$\Gamma'(1) = \int_0^\infty e^{-x}\log x\cdot dx = -\gamma = -0.5772157\ldots \tag{1.30}$$

by which the Euler constant γ is defined.

Finally from the theory of the asymptotic approximations of the gamma-function we shall need the so-called Stirling's formula in the form

$$\log\Gamma(s+b) = (s+b-1/2)\log s - s + \tfrac{1}{2}\log 2\pi + O(|s|^{-1}), \tag{1.31}$$

which is valid for b constant and $|\arg s| \le \pi - \delta$ $(\delta > 0)$, if $s = 0$ and neighborhoods of poles of $\Gamma(s+b)$ are excluded. Also we have

$$|\Gamma(s)| = (2\pi)^{1/2}|t|^{\sigma-1/2}e^{-\pi|t|/2}(1 + O(|t|^{-1})), \quad t \ge t_0, \tag{1.32}$$

valid for $C_1 \le \sigma \le C_2$, and the O-constant depending on C_1, C_2; and for $\delta > 0$ fixed, $|\arg s| \le \pi - \delta$, $|s| \ge \delta$ we have

$$\Gamma'(s)/\Gamma(s) = \log s - 1/(2s) + O(|s|^{-2}). \tag{1.33}$$

§7. An exponential integral

Very often we shall smoothen integrals by introducing a certain exponential weight which simplifies subsequent estimations. The integral that is needed is

$$\int_{-\infty}^{\infty}\exp(At - Bt^2)dt = (\pi/B)^{1/2}\exp(A^2/4B), \quad (\mathrm{Re}\ B > 0) \tag{1.34}$$

which in fact represents an analytic function of A and B provided that Re B > 0. By analytic continuation it is sufficient to prove (1.34) for B real and positive, when the change of variable $t = A/(2B) + xB^{-1/2}$ gives

$$\int_{-\infty}^{\infty} \exp(At - Bt^2)dt = B^{-1/2}\exp(A^2/4B)\int_{-\infty}^{\infty} e^{-x^2}dx = (\pi/B)^{1/2}\exp(A^2/4B).$$

§8. The Halász-Montgomery inequalities

The inequalities in question are certain general inequalities for vectors in inner-product spaces which have found many applications recently in analytic number theory. Their connection with large sieve inequalities is very close, and the whole subject is extensively treated by H.L. Montgomery [2], [5], [6], where detailed references are given. To formulate the inequalities, suppose that $\xi, \varphi_1, \ldots, \varphi_R$ are arbitrary vectors in an inner product vector space over C, where (a,b) will be the notation for the inner product and $\|a\|^2 = (a,a)$. Then

$$\sum_{r\le R} |(\xi,\varphi_r)| \le \|\xi\|\Big(\sum_{r,s\le R} |(\varphi_r,\varphi_s)|\Big)^{1/2}, \tag{1.35}$$

$$\sum_{r\le R} |(\xi,\varphi_r)|^2 \le \|\xi\|^2 \max_{r\le R} \sum_{s\le R} |(\varphi_r,\varphi_s)|. \tag{1.36}$$

Both of these inequalities are derived by judicious use of the Cauchy-Schwarz inequality for vector spaces. To see this observe that from $(a,b) = \overline{(b,a)}$ one has

$$\sum_{r\le R} c_r(\xi,\varphi_r) = \Big(\xi, \sum_{r\le R} \bar{c}_r \varphi_r\Big)$$

for any scalars c_r. Thus

$$\Big|\sum_{r\le R} c_r(\xi,\varphi_r)\Big|^2 \le \|\xi\|^2 \Big\|\sum_{r\le R} \bar{c}_r\varphi_r\Big\|^2 = \|\xi\|^2 \sum_{r,s\le R} \bar{c}_r c_s(\varphi_r,\varphi_s). \tag{1.37}$$

If we take $c_r = \exp(-i\arg(\xi,\varphi_r))$, then $|c_r| = 1$ and

$$\sum_{r\le R} c_r(\xi,\varphi_r) = \sum_{r\le R} |(\xi,\varphi_r)|,$$

so that (1.35) follows at once from (1.37). For (1.36) we use the elementary inequality

$$|\bar{c}_r c_s| \le \tfrac{1}{2}|c_r|^2 + \tfrac{1}{2}|c_s|^2$$

to obtain

$$(1.38) \qquad \sum_{r,s\leq R} \bar{c}_r c_s (\varphi_r, \varphi_s) \leq \sum_{r\leq R} |c_r|^2 \sum_{s\leq R} |(\varphi_r, \varphi_s)| \leq \sum_{r\leq R} |c_r|^2 \max_{r\leq R} \sum_{s\leq R} |(\varphi_r, \varphi_s)|,$$

so that combining (1.37) and (1.38) we have (1.36) if we take $c_r = \overline{(\xi, \varphi_r)}$.

If $a = \{a_n\}_{n=1}^{\infty}$ and $b = \{b_n\}_{n=1}^{\infty}$ are two (vector) sequences of complex numbers, then the standard inner product of a and b is defined as

$$(1.39) \qquad (a,b) = \sum_{n=1}^{\infty} a_n \bar{b}_n.$$

Aleksandar Ivić

TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 2

EXPONENTIAL INTEGRALS AND EXPONENTIAL SUMS

CHAPTER 2

EXPONENTIAL INTEGRALS AND EXPONENT PAIRS

§1. Exponential integrals

The topic of this chapter involves one of the very important and difficult parts of analytic number theory. Exponential integrals and exponential sums occur in a large number of problems whose solutions ultimately depend on asymptotic formulas or good O-bounds for these integrals or sums. A deep method for dealing with exponential sums and integrals has been founded by J.G. van der Corput [1], [2] in the 1920's. This is the so-called "saddle point method" or the "method of the stationary phase", which has much advanced analytic number theory and brought on remarkable improvements in many classical problems such as divisor problems, circle problem, order of the zeta-function in the critical strip etc. This method is systematized here in the theory of (one-dimensional) exponent pairs which will be presented here in a simplified form, due mostly to E. Phillips [1]. Albeit the theory of exponent pairs is, in general, superseded by two-dimensional and multi-dimensional methods, this theory is nevertheless fairly simple to use in practice. Furthermore the best existing multi-dimensional theory of exponential integrals and sums, due to G. Kolesnik in his series of papers [1], [2], [3], [5] and [6], is both very difficult and has not produced so far dramatic improvements over results obtainable by the classical theory of exponent pairs. Therefore we shall restrict ourselves to the classical theory of exponent pairs, devoting this section to the estimation of certain exponential integrals. The main results will be stated as theorems, while other results will be given as lemmas. We begin with

Lemma 2.1. Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic and $F'(x) \geq m > 0$ or $F'(x) \leq -m < 0$ for $a \leq x \leq b$. Then

$$\left|\int_a^b e^{iF(x)}dx\right| \leq 4m^{-1}. \tag{2.1}$$

Proof of Lemma 2.1. Since the conjugate of $\int_a^b e^{iF(x)}dx$ is $\int_a^b e^{-iF(x)}dx$, this means that in most problems involving exponential sums and integrals we may suppose that $F'(x) > 0$. Now $e^{iF(x)} = \cos F(x) + i\sin F(x)$, and by the second mean value theorem for real integrals

$$(2.2) \qquad \int_a^b f(x)g(x)dx = \begin{cases} f(b)\int_c^b g(x)dx, & \text{if } f(x) \geq 0,\ f'(x) \geq 0,\ x \in [a,b] \\ f(a)\int_a^c g(x)dx, & \text{if } f(x) \geq 0,\ f'(x) \leq 0,\ x \in [a,b] \end{cases}$$

where c is some number satisfying $a < c < b$. Writing

$$\int_a^b \cos F(x).dx = \int_a^b (F'(x))^{-1} d(\sin F(x))$$

and using (2.2) it is seen that

$$\left|\int_a^b \cos F(x)\cdot dx\right| \leq 2m^{-1},$$

since $(F'(x))^{-1}$ is monotonic in $[a,b]$ because $(1/F'(x))' = -F''(x)/(F'(x))^2$ and $F''(x)$ is of constant sign since $F'(x)$ is monotonic in $[a,b]$. The same bound holds also for the integral with $\sin F(x)$, hence (2.1). Using (2.2) and the same argument it is also seen that

$$(2.3) \qquad \left|\int_a^b G(x)e^{iF(x)}dx\right| \leq 4Gm^{-1},$$

where F is as in Lemma 2.1, and $G(x)$ is a positive, monotonic function for $a \leq x \leq b$ such that $|G(x)| \leq G$.

<u>Lemma 2.2.</u> Let $F(x)$ be a twice differentiable function in $[a,b]$ such that $F''(x) \geq m > 0$ or $F''(x) \leq -m < 0$. Then

$$(2.4) \qquad \left|\int_a^b e^{iF(x)}dx\right| \leq 8m^{-1/2}.$$

<u>Proof of Lemma 2.2.</u> Assume that $F''(x) > 0$, so that $F'(x)$ is monotonically increasing and has at most one zero c, i.e. $F'(c) = 0$ with $a < c < b$. Write

$$\int_a^b e^{iF(x)}dx = \int_a^{c-u} + \int_{c-u}^{c+u} + \int_{c+u}^b = I_1 + I_2 + I_3,$$

say, where $u > 0$ will be suitably determined. Trivially $|I_2| \leq 2u$, and for $u < c - a$ and $a \leq x \leq c - u$ we have $|F'(x)| = \left|\int_x^c F''(t)dt\right| \geq um$, so that Lemma 2.1 gives

$|I_1| \leq 4(um)^{-1}$ and a similar estimate holds for I_3 if $u < b - c$. If $u \geq c - a$ or $u \geq b - c$, or if F' has no zero in $[a,b]$, the analysis is similar, and in all cases leads to

$$\left| \int_a^b e^{iF(x)} dx \right| \leq 8(um)^{-1} + 2u = 8m^{-1/2}$$

if we take $u = 2m^{-1/2}$. Analogously to (2.3) one has

$$\left| \int_a^b G(x) e^{iF(x)} dx \right| \leq 8Gm^{-1/2} \tag{2.5}$$

if F satisfies the hypothesis of Lemma 2.2, and $G(x)$ is a positive, monotonic function for $a \leq x \leq b$ such that $|G(x)| \leq G$.

Lemmas 2.1 and 2.2 are very general, but they have the shortcoming that the estimates given for exponential integrals are only upper bounds which do not explicitly depend on the length of the interval of integration. We present now a "saddle point" theorem, which shows that the main contribution to the exponential integral comes from its saddle point (i.e. the point where the first derivative $F'(x)$ in $e^{iF(x)}$ vanishes), provided that certain conditions are satisfied. This is

<u>THEOREM 2.1.</u> Suppose that $f(x)$ is a real-valued function such that $f(x) \in C^4[a,b]$, $f''(x) < 0$ for $x \in [a,b]$ and

$$m_2 \asymp |f''(x)|, \quad |f^{(3)}(x)| \ll m_3, \quad |f^{(4)}(x)| \ll m_4,$$

where $m_3^2 = m_2 m_4$. Then if $f'(c) = 0$ for $a \leq c \leq b$ we have

$$\int_a^b e(f(x))dx = e(f(c) - 1/8)|f''(x)|^{-1/2} + O(m_2^{-1} m_3^{1/3}) + \tag{2.6}$$

$$+ O(\min(m_2^{-1/2}, |f'(a)|^{-1})) + O(\min(m_2^{-1/2}, |f'(b)|^{-1})).$$

If $f''(x) > 0$ in $[a,b]$ and the other hypotheses hold, then the same result is obtained with $e(f(c) + 1/8)$ in place of $e(f(c) - 1/8)$.

<u>Proof of Theorem 2.1.</u> The cases $f'' < 0$ and $f'' > 0$ are analogous, so only the former is considered. Write

$$\int_a^b e(f(x))dx = \int_a^{c-u} e(f(x))dx + \int_{c-u}^{c+u} e(f(x))dx + \int_{c+u}^b e(f(x))dx = I_1 + I_2 + I_3, \tag{2.7}$$

say, where we suppose that u satisfies $u \leq \min(c - a, b - c)$ and will be suitably determined later. By Lemma 2.1

$$|I_1| \ll 1/|f'(c-u)| = 1/\left|\int_c^{c-u} f''(t)dt\right| \ll (um_2)^{-1},$$

and similarly the same bound holds also for I_3.

Since $f'(c) = 0$ we use Taylor's formula to obtain with some θ for which $|\theta| \le 1$

$$I_2 = \int_{-u}^{u} e(f(x+c))dx = \int_{-u}^{u} e(f(c) + \frac{x^2}{2!}f''(c) + \frac{x^3}{3!}f^{(3)}(c) + \frac{x^4}{4!}f^{(4)}(c+\theta x))dx =$$

(2.8) $$= e(f(c)) \int_{-u}^{u} e(\tfrac{1}{2}x^2 f''(c) + \tfrac{1}{6}x^3 f^{(3)}(c))\cdot(1 + O(|x|^4 m_4))dx =$$

$$= e(f(c)) \int_{-u}^{u} e(\tfrac{1}{2}x^2 f''(c) + \tfrac{1}{6}x^3 f^{(3)}(c))dx + O(u^5 m_4).$$

Abbreviating $F = \frac{1}{3}\pi i f^{(3)}(c)$, the last integral in (2.8) becomes

(2.9) $$\int_{-u}^{u} e(\tfrac{1}{2}x^2 f''(c))\exp(Fx^3)dx = 2\int_0^{u} e^{\pi i f''(c)x^2}(1 + \sum_{r=1}^{\infty}\frac{(Fx^3)^{2r}}{(2r)!})dx,$$

since the integrals involving odd powers of r vanish identically. Making the change of variable $\pi|f''(c)|x^2 = y$ in the integrals appearing on the right-hand side of (2.9) we obtain

(2.10) $$(\pi|f''(c)|)^{-1/2}\int_0^{\pi|f''(c)|u^2} e^{-iy}y^{-1/2}dy + \sum_{r=1}^{\infty}\frac{F^{2r}}{(2r)!}\int_0^{\pi|f''(c)|u^2} e^{-iy}y^{3r-1/2}(\pi|f''(c)|)^{-3r-1/2}dy.$$

Applying Cauchy's integral theorem to the function $\exp(-iz^2)$ and the sector of the circle of radius $x^{1/2}$, center at the origin and endpoints $z_1 = x^{1/2}$, $z_2 = e(-1/8)x^{1/2}$, we obtain

$$\int_0^{x} e^{-iy}y^{-1/2}dy = 2\int_0^{x^{1/2}} e^{-iz^2}dz = \pi^{1/2}e^{-\pi i/4} + O(x^{-1/2}),$$

if we use $\int_0^{\infty}\exp(-z^2)dz = \frac{1}{2}\pi^{1/2}$. Therefore the first term in (2.10) is

$$e(-1/8)|f''(c)|^{-1/2} + O((um_2)^{-1}),$$

and the remaining terms are by (2.2)

$$\ll \sum_{r=1}^{\infty} \frac{F^{2r}}{(2r)!} m_2^{-3r-1/2}(m_2u^2)^{3r-1/2} = (um_2)^{-1}\sum_{r=1}^{\infty}\frac{(Fu^3)^{2r}}{(2r)!} \ll (um_2)^{-1}\exp(cu^3m_3)$$

for some absolute $c > 0$. Therefore we obtain

$$(2.11)\quad \int_a^b e(f(x))dx = e(f(c)-1/8)|f''(c)|^{-1/2} + O(u^{-1}m_2^{-1}) + O(u^5m_4) + $$
$$+ O(u^{-1}m_2^{-1}\exp(cu^3m_3)),$$

and choosing $u = (m_2m_4)^{-1/6} = (m_3)^{-1/3}$ the error terms above are of the same order of magnitude. This proves then (2.6) if $u \le \min(c - a, b - c)$. If this condition is not satisfied suppose first that $b - u < c \le b$. Proceeding as above it is seen that there is an extra error term

$$(2.12)\quad \begin{aligned} I_4 &= e(f(c))\int_{b-c}^{u} e(\tfrac{1}{2}x^2f''(c) + \tfrac{1}{6}x^3f^{(3)}(c))dx = \\ &e(f(c))\int_{b-c}^{u} e(\tfrac{1}{2}x^2f''(c))dx + \sum_{r=1}^{\infty}\frac{F^r}{r!}\int_{b-c}^{u} e(\tfrac{1}{2}x^2f''(c))x^{3r}dx \end{aligned}$$

to be dealt with, where as before $F = \frac{1}{3}\pi i f^{(3)}(c)$. By (2.3) we have uniformly in r

$$(2.13)\quad \int_{b-c}^{u} e(\tfrac{1}{2}x^2f''(c))x^{3r}dx \ll u^{3r-1}m_2^{-1},$$

while Lemma 2.1 and Lemma 2.2 yield

$$(2.14)\quad \int_{b-c}^{u} e(\tfrac{1}{2}x^2f''(c))dx \ll \min(|f'(b)|^{-1}, m_2^{-1/2}),$$

since for $F(x) = \frac{1}{2}f''(c)x^2$ and $a \le x \le b$

$$|F'(x)| = x|f''(c)| \gg (b - c)m_2 \gg |f'(b)|,$$

because by the mean value theorem for some $c < \xi < b$ we have

$$f'(b) = f'(b) - f'(c) = (b - c)f''(\xi) \ll (b - c)m_2.$$

Therefore (2.13) and (2.14) give for I_4 in (2.12)

$$I_4 \ll \min(|f'(b)|^{-1}, m_2^{-1/2}) + \sum_{r=1}^{\infty}(um_2)^{-1}\frac{(Fu^3)^r}{r!} \ll$$
$$\ll \min(|f'(b)|^{-1}, m_2^{-1/2}) + (um_2)^{-1}\exp(cu^3m_3),$$

and (2.6) follows again with $u = m_3^{-1/3}$. Similarly $O(\min(|f'(a)|^{-1}, m_2^{-1/2}))$ appears in (2.6) if $c - u < a$, and (2.6) follows in all cases.

Theorem 2.1 is sharp when m_2 (i.e. the order of f") is sufficiently large, and enough additional information about $f^{(3)}$ and $f^{(4)}$ is known. On the weaker assumption that $m_2 \asymp f''(x)$, $f^{(3)}(x) \ll m_3$ for $a \le x \le b$, we can obtain by the method of proof of Theorem 2.1

$$\int_a^b e(f(x))dx = e(f(c)-1/8)\,|f''(c)|^{-1/2} + O(m_2^{-4/5} m_3^{1/5}) + \tag{2.15}$$

$$+ O(\min(|f'(a)|^{-1}, m_2^{-1/2})) + O(\min(|f'(b)|^{-1}, m_2^{-1/2})),$$

where $f'(c) = 0$ and $f'' < 0$, and if $f'' > 0$ then $e(f(c)-1/8)$ is to be replaced by $e(f(c)+1/8)$. The proof of (2.15) (where no information about $f^{(4)}$ is needed) is easier than the proof of (2.6), since for (2.15) only the first three terms in Taylor's formula for I_2 are taken, while for (2.6) we needed the first four terms in Taylor's formula.

Next we shall formulate and prove another result which is similar to Theorem 2.1. The main difference will be that instead of $f(x)$ we consider $f(z)$, where z is a complex variable lying in a suitable domain, and suppose that $f(z)$ is real when z is real and lies in $[a,b]$. The main term will turn out to be essentially the same one as in (2.6), but the error terms will be different and in certain applications sharper than those in (2.6). The result is

<u>THEOREM 2.2.</u> Let $f(z)$, $\varphi(z)$ be two functions of the complex variable z and $[a,b]$ a real interval such that

i) for $a \le x \le b$ the function $f(x)$ is real and $f''(x) > 0$;

ii) for a certain positive differentiable function $\mu(x)$, defined on $a \le x \le b$, $f(z)$ and $\varphi(z)$ are analytic for $a \le x \le b$, $|z - x| \le \mu(x)$;

iii) there exist positive functions $F(x), \Phi(x)$ defined on $[a,b]$ such that for $a \le x \le b$, $|z - x| \le \mu(x)$ we have

$$\varphi(z) \ll \Phi(x),\quad f'(z) \ll F(x)\mu^{-1}(x),\quad |f''(z)|^{-1} \ll \mu^2(x)F^{-1}(x),$$

and the $\ll$-constants are absolute.

Let k be any real number, and if $f'(x) + k$ has a zero in $[a,b]$ denote it by x_0. Let the values of $f(x)$, $\varphi(x)$ etc. at a, x_0, b be characterized by the suffixes a,o and b respectively. Then

$$\int_a^b \varphi(x)e(f(x)+kx)dx = \varphi_o f_o''^{-1/2}e(f_o+kx_o+1/8) +$$

(2.16) $$O\left(\int_a^b \Phi(x)\exp(-C|k|\mu(x)-CF(x))(dx+|d\mu(x)|)\right) + O(\Phi_o\mu_o F_o^{-3/2}) +$$

$$O(\Phi_a(|f_a'+k| + f_a''^{1/2})^{-1}) + O(\Phi_b(|f_b'+k|+f_b''^{1/2})^{-1}).$$

If $f'(x)+k$ has no zero for $a \le x \le b$, then the terms involving x_o are to be omitted.

<u>Proof of Theorem 2.2</u>. We shall consider only the more difficult case when $f'(x)+k$ has a zero in $[a,b]$, and we shall denote $\lambda(x) = \alpha\mu(x)$ for a suitable $0 < \alpha < 1/2$ to be determined later. As in the proof of Theorem 2.1 we shall split the integral on the left-hand side of (2.16) into several integrals. By Cauchy's integral theorem we can replace the path of integration by the contour joining the points $a, a-\lambda_a(1+i), x_o-\lambda_o(1+i), x_o+\lambda_o(1+i), b+\lambda_b(1+i), b$. Denoting the corresponding integrals by $I_1,\dots,I_5$ respectively, we take I_1, I_3 and I_5 along straight lines, and I_2 and I_4 are to be taken along the loci of the points $x \pm \lambda(x)(1+i)$ respectively.

Therefore for $z = x+(1+i)y$, $-\lambda(x) \le y \le \lambda(x)$, $a \le x \le b$ we have

$$f(z)+kz = f(x)+kx+(1+i)y(f'(x)+k)+iy^2f''(x)+\theta(y),$$

where by Taylor's formula

$$\theta(x) \ll F(x)\sum_{n=3}^{\infty}|z-x|^n|f^{(n)}(x)|F^{-1}(x)/n! \ll F(x)|y|^3\mu^{-3}(x),$$

since by iii) and Cauchy's formula for derivatives of analytic functions

$$f^{(n)}(x) \ll F(x)\mu^{-n}(x),\ n = 2,3,\dots,$$

Hence by taking α sufficiently small we obtain $|\theta(y)| < \frac{1}{2}y^2f''(x)$, which gives

(2.17) $$\mathrm{Re}(2\pi i(f(z)+kz)) < -2\pi y(f'(x)+k) - \pi y^2 f''(x),$$

(2.18) $$I_1 \ll \int_0^{\lambda_a}\Phi_a\exp(-2\pi y|f_a'+k| - \pi y^2 f_a'')dy \ll \Phi_a(|f_a'+k|+f_a''^{1/2})^{-1},$$

and a similar bound holds for I_5 with the suffix a replaced by b. By the same argument

(2.19) $$I_2 \ll \int_a^{x_a} \phi(x)\exp(-2\pi\lambda(x)\,|f'(x)+k| - \pi\lambda^2(x)f''(x))(dx + |d\lambda(x)|).$$

Now if $|k| \le 2|f'(x)|$, then by iii)

$$\lambda(x)\,|f'(x)+k| \ll \lambda(x)\,|f'(x)| \ll F(x),\quad k\mu(x) \ll f'(x)\mu(x) \ll F(x),$$

while for $|k| > 2|f'(x)|$

$$\lambda(x)\,|f'(x)+k| \ge \lambda(x)(|k| - |f'(x)|) \gg |k|\mu(x),$$

and since $\lambda^2(x)f''(x) \gg F(x)$ by iii), it is seen that in any event

$$-2\pi\lambda(x)\,|f'(x) + k| - \pi\lambda^2(x)f''(x) < -C|k|\mu(x) - CF(x).$$

This gives

(2.20) $$I_2 \ll \int_a^{x_a} \phi(x)\exp(-C\,|k|\mu(x) - CF(x))(dx + |d\mu(x)|),$$

and a corresponding bound for I_4.

From (2.18) and (2.20) it is seen that it remains yet to show

(2.21) $$I_3 = \varphi_0 f_0''^{-1/2} e(f_0 + kx_0 + 1/8) + O(\phi_0\mu_0 F_0^{-3/2}),$$

and then the proof will be finished. To accomplish this write

$$I_3 = \int_{-\lambda_0(1+i)}^{\lambda_0(1+i)} \varphi(x_0+y)e(f(x_0+y) + kx_0 + ky)dy =$$

$$= \int_{-\lambda_0(1+i)}^{-v(1+i)} + \int_{-v(1+i)}^{v(1+i)} + \int_{v(1+i)}^{\lambda_0(1+i)} = I_{31} + I_{32} + I_{33},$$

say, where we choose

(2.22) $$v = \lambda_0(1 + F_0^{1/3})^{-1}.$$

The integrals I_{31} and I_{33} are estimated analogously and yield the error term in (2.21), while the main term in (2.21) will come from I_{32}. By (2.17) and the change of variable $\pi y^2 f_0'' = x$ we have

(2.23) $$I_{33} \ll \phi_0 \int_v^\infty e^{-\pi y^2 f_0''}dy \ll \phi_0 v^{-1} f_0''^{-1} \int_{\pi v^2 f_0''}^\infty e^{-x}dx \ll \phi_0 (vf_0'')^{-1} e^{-\pi v^2 f_0''}.$$

Hence

$$v^2 f_0'' = \lambda_0^2 f_0''(1 + F_0^{1/3})^{-2} \gg F_0(1 + F_0^{1/3})^{-2}$$

by iii), and also

$$(vf_0'')^{-1} = v(v^2 f_0'')^{-1} \ll \mu_0 F_0^{-1}(1 + F_0^{1/3}).$$

Therefore for $F_o \geq 1$

$$I_{33} \ll \Phi_o \mu_o F_o^{-2/3} \exp(-CF_o^{1/3}),$$

while for $F_o < 1$

$$I_{33} \ll \Phi_o \mu_o F_o^{-1},$$

so that in any case

$$I_{31} + I_{33} \ll \Phi_o \mu_o F_o^{-3/2}. \tag{2.24}$$

The estimation of I_{32} bears resemblance to the estimation of I_2 in (2.8). In both cases Taylor's formula is used and the fact that the first derivative vanishes at a certain point ("saddle point"). In I_{32} we write

$$e(f(x_o+y)+k(x_o+y)) = e(f_o+f_o'y+\frac{1}{2}f_o''y^2 + \frac{1}{6}f_o^{(3)}y^3 + \sum_{r=4}^{\infty}\frac{f_o^{(r)}y^r}{r!} + kx_o + ky). \tag{2.25}$$

Now by hypothesis $f'(x_o) + k = f_o' + k = 0$, and using $f^{(r)}(x) \ll \mu^{-r}(x)F(x)$ and $e^{iu} = 1 + iu + O(u^2)$ (u real), we see that the left-hand side of (2.25) is equal to

$$e(f_o + kx_o + \frac{1}{2}f_o''y^2)\cdot\left\{1 + \frac{1}{3}\pi i f_o^{(3)}y^3 + O(y^6F_o^2\mu_o^{-6}) + O(y^4F_o\mu_o^{-4})\right\}. \tag{2.26}$$

Next by a change of variable $y = (1+i)Y$ we have

$$\int_{-v(1+i)}^{v(1+i)} y^{2k}e^{i\pi y^2 f_o''}dy = \int_{-v}^{v} Y^{2k}e^{-2\pi Y^2 f_o''}(1+i)^{2k+1}dY \ll (f_o'')^{-k-1/2} \ll F_o^{-k-1/2}\mu_o^{2k+1}$$

for $k > 0$ a fixed integer, so that the contribution of the error terms in (2.26) to I_{32} will be (since the integrals with odd powers vanish)

$$\ll \Phi_o \mu_o F_o^{-3/2}. \tag{2.27}$$

As regards the terms that remain in (2.26) we have (in view of $\varphi_o' \ll \Phi_o \mu_o^{-1}$)

$$\varphi(x_o+y)(1 + \frac{1}{6}y^3 f_o^{(3)}) = \varphi_o + y\varphi_o' + \frac{1}{6}y^3\varphi_o f_o^{(3)} + O(y^2\Phi_o\mu_o^{-2}) + $$

$$+ O(y^4\Phi_o F_o \mu_o^{-4}).$$

Arguing as before it is seen that we are left with

$$\int_{-v(1+i)}^{v(1+i)} \varphi(x_o+y)(1 + \frac{1}{6}y^3 f_o^{(3)})e^{\pi i y^2 f_o''}dy = \varphi_o \int_{-v(1+i)}^{v(1+i)} e^{\pi i y^2 f_o''}dy + \tag{2.28}$$

$$+ O(\Phi_o\mu_o F_o^{-3/2}) + O(\Phi_o F_o \mu_o^{-4} f_o''^{-5/2}) = \varphi_o \int_{-\infty(1+i)}^{\infty(1+i)} e^{\pi i y^2 f_o''}dy +$$

$$+ \; O\Big(\varphi_o\Big|\int_{v(1+i)}^{\infty(1+i)} e^{\pi i y^2 f''_o} dy\Big|\Big) + O(\Phi_o\mu_o F_o^{-3/2}) = \varphi_o (f''_o)^{-1/2} e^{\pi i/4} + O(\Phi_o \mu_o F_o^{-3/2}),$$

since by (1.34) and $y = (1+i)Y$

$$\int_{-\infty(1+i)}^{\infty(1+i)} e^{\pi i y^2 f''_o} dy = (1+i)\int_{-\infty}^{\infty} e^{-2\pi Y^2 f''_o} dY = (1+i)(\pi/2\pi f''_o)^{1/2} = (f''_o)^{-1/2} e(1/8),$$

and where as in (2.23)

$$\int_{v(1+i)}^{\infty(1+i)} e^{\pi i y^2 f''_o} dy \ll \mu_o F_o^{-3/2}.$$

This completes the proof of (2.16) in case $f'(x) + k$ vanishes in $[a,b]$. In the other case we take the contour of integration as $a, a \pm \lambda_a(1+i), b \pm \lambda_b(1+i), b$ depending on whether $f'(x)+k \geq 0$ or ≤ 0 in $[a,b]$, and then there is no term corresponding to I_3. Also similarly as in Theorem 2.1, if all the hypotheses of Theorem 2.2 hold but $f''(x) < 0$ in $[a,b]$, then the main term in (2.16) is

$$\varphi_o |f''_o|^{-1/2} e(f_o + kx_o - 1/8).$$

Theorem 2.1 is essentially the same as Theorem 2.2 with $f(x)$ instead of $f(x) + k$, $\varphi(x) \equiv 1$ and different hypotheses on f which lead to different error terms.

We end this section by presenting a lemma which involves double exponential integrals with no saddle point. This is

<u>Lemma 2.3</u>. Let $f(z)$ and $g(z)$ be two functions of the complex variable z such that

i) $f(x)$ is real for $a \leq x \leq b$;

ii) $f(z)$ and $g(z)$ are analytic for $|z - x| \leq \mu$ for some $\mu > 0$ and some $x \in [a,b]$;

iii) $g(z) \ll G$, $|f'(z)| \asymp M$ for $|z - x| \leq \mu$.

Let $0 < U < \frac{1}{2}(b - a)$. Then for some absolute $A > 0$

$$U^{-1}\int_0^U \Big(\int_{a+u}^{b-u} g(x) e(f(x)) dx\Big) du \ll G e^{-A\mu M}(b - a + \mu) + GM^{-2}U^{-1}. \tag{2.29}$$

<u>Proof of Lemma 2.3</u>. By iii) and continuity $f'(x)$ is of the same sign, say positive, in $[a,b]$. Let $C(u)$ denote the contour of segments joining the points

$a+u, a+u+i\alpha\mu$, $b-u+i\alpha\mu$, $b-u$, where $0<\alpha<1/2$ is a number which will be specified in a moment. By iii) and Cauchy's formula for derivatives of analytic functions we have $f^{(n)}(z) \ll M\mu^{1-n}$ for $n \geq 2$, and hence by Taylor's formula for $z = x+iy \in C(u)$

$$f(z) = f(x) + iyf'(x) + \theta(x,y);\ \theta(x,y) \ll \sum_{n=2}^{\infty} |z-x|^n |f^{(n)}(x)|/n! \ll My^2\mu^{-1},$$

and so

$$\text{Im } f(x+iy) \gg My, \quad (x+iy \in C(u)) \tag{2.30}$$

if α is chosen sufficiently small, since $|y| \leq \alpha\mu$ for $z \in C(u)$.

By Cauchy's integral theorem

$$U^{-1}\int_0^U \int_{a+u}^{b-u} g(x)e(f(x))dxdu = U^{-1}\int_0^U \Big(\int_{C(u)} g(z)e(f(z))dz\Big)du. \tag{2.31}$$

In view of (2.30) the integral over the horizontal side of $C(u)$ is

$$\int_{a+u+i\alpha\mu}^{b-u+i\alpha\mu} g(z)e(f(z))dz \ll (b-a)Ge^{-A\mu M}$$

uniformly in u, with some absolute $A > 0$. For the vertical side joining $a+u$ and $a+u+i\alpha\mu$ we have

$$\Big|U^{-1}\int_0^U \Big(\int_{a+u}^{a+u+i\alpha\mu} g(z)e(f(z))dz\Big)du\Big| = \Big|U^{-1}\int_0^{\alpha\mu}\Big(\int_{a+iy}^{a+U+iy} g(z)e(f(z))dz\Big)dy\Big|, \tag{2.32}$$

if we write $z = x+iy = a+u+iy, dz = idy$, $0 \leq y \leq \alpha\mu$ and invert the order of integration. An application of Cauchy's integral theorem to the rectangle with vertices $a+iy, a+i\alpha\mu, a+U+i\alpha\mu, a+U+iy$ gives in view of iii) and (2.30) that

$$\int_{a+iy}^{a+U+iy} g(z)e(f(z))dz \ll G\Big(\int_y^{\infty} e^{-AMv}dv + Ue^{-A\mu M}\Big) \ll G(M^{-1}e^{-AMy} + Ue^{-A\mu M}),$$

and therefore the left-hand side of (2.32) is

$$\ll U^{-1}\int_0^{\alpha\mu} G(M^{-1}e^{-AMy} + Ue^{-A\mu M})dy \ll GM^{-2}U^{-1} + G\mu e^{-A\mu M}.$$

A similar estimate can be obtained for the vertical side joining $b-u+i\alpha\mu$

and b - u, and the case $f'(x) < 0$ is dealt with analogously by taking the contour in the lower half-plane. This proves (2.29).

§2. Exponential sums

By exponential sums we shall mean here sums of the type $\sum_{a<n\leq b} e(f(n))$, where $f(x)$ is real for $a \leq x \leq b$ and f is (suffiviently many times) differentiable. We begin with a result which transforms an exponential sum into a sum of exponential integrals, which are easier to estimate in view of the results of the preceding section. This is

Lemma 2.4. Let $f(x)$ be a real function for $a \leq x \leq b$ such that $f(x) \in C^2[a,b]$ and $f''(x) < 0$ in $[a,b]$ and let $f'(b) = \alpha, f'(a) = \beta$. Then for $0 < \eta < 1$ arbitrary we have

$$\sum_{a<n\leq b} e(f(n)) = \sum_{\alpha-\eta<m<\beta+\eta} \int_a^b e(f(x) - mx)dx + O(\log(\beta - \alpha + 2)). \tag{2.33}$$

Proof of Lemma 2.4. By the Euler-Maclaurin summation formula (1.21)

$$\sum_{a<n\leq b} e(f(n)) = \int_a^b e(f(x))dx + 2\pi i \int_a^b \psi(x)f'(x)e(f(x))dx + O(1), \tag{2.34}$$

where $\psi(x) = x - [x] - 1/2$. Without loss of generality we may suppose $\eta - 1 < \alpha \leq \eta$ (so that $m \geq 0$), for if k is an integer such that $\eta - 1 < \alpha - k \leq \eta$, then (2.33) becomes with $h(x) = f(x) - kx$

$$\sum_{a<n\leq b} e(f(n)) = \sum_{a<n\leq b} e(h(n)) = \sum_{\alpha'-\eta<m-k<\beta'+\eta} \int_a^b e(h(x) - (m-k)x)dx + \tag{2.35}$$

$$+ O(\log(\beta' - \alpha' + 2)),$$

where $\alpha' = \alpha - k$, $\beta' = \beta - k$, so that (2.35) implies (2.33), and $m - k \geq 0$ by the choice of k. Using the Fourier expansion (1.25) for $\psi(x)$ it is seen that the second integral in (2.34) is equal to

$$-2i\sum_{m=1}^{\infty} \int_a^b m^{-1}\sin 2\pi mx\cdot e(f(x))f'(x)dx = \sum_{m=1}^{\infty} m^{-1}\int_a^b (e(-mx)-e(mx))e(f(x))f'(x)dx =$$

$$\sum_{m=1}^{\infty}(2\pi i m)^{-1}\int_a^b \frac{f'(x)}{f'(x)-m}d(e(f(x)-mx)) - \sum_{m=1}^{\infty}(2\pi i m)^{-1}\int_a^b \frac{f'(x)}{f'(x)+m}d(e(f(x)+mx)).$$

By hypothesis $f'(x)$ is monotonically decreasing for $a \le x \le b$, and so is then also $f'(x)/(f'(x)+m)$. An application of (2.2) to the second integral above shows that it is $\ll \beta/(\beta+m)$ uniformly in m, so that the whole sum is

$$\ll \sum_{m=1}^{\infty}\beta m^{-1}(\beta+m)^{-1} \ll \sum_{m\le\beta} m^{-1} + \sum_{m>\beta}\beta m^{-2} \ll 1 + \log(\beta+2).$$

Similarly it is seen that

$$\sum_{m\ge\beta+\eta} m^{-1}\int_a^b \frac{f'(x)}{f'(x)-m}d(e(f(x)-mx)) \ll \sum_{m\ge\beta+\eta} m^{-1}(m-\beta)^{-1}\beta \ll$$

$$\sum_{\beta+\eta\le m\le 2\beta}(m-\beta)^{-1} + \sum_{m\ge 2\beta}\beta m^{-2} \ll 1 + \log(\beta+2).$$

It remains yet to estimate

$$\sum_{1\le m<\beta+\eta}(2\pi i m)^{-1}\int_a^b \frac{f'(x)}{f'(x)-m}d(e(f(x)-mx)) = \sum_{1\le m<\beta+\eta} m^{-1}\int_a^b f'(x)e(f(x)-mx)dx =$$

$$(2\pi i)^{-1}\sum_{1\le m<\beta+\eta} m^{-1}\int_a^b d(e(f(x)-mx)) + \sum_{1\le m<\beta+\eta}\int_a^b e(f(x)-mx)dx =$$

$$O(\log(\beta+\eta)) + \sum_{1\le m<\beta+\eta}\int_a^b e(f(x)-mx)dx.$$

Taking into account the first integral on the right-hand side of (2.34) we finally obtain

$$\sum_{a<n\le b} e(f(n)) = \sum_{0\le m<\beta+\eta}\int_a^b e(f(x)-mx)dx + O(\log(\beta+2)),$$

which by the discussion made concerning (2.35) proves (2.33).

<u>Lemma 2.5</u>. Let $f(x)$ be a real differentiable function of x for $a \le x \le b$ such that $f'(x)$ is monotonic and $|f'(x)| \le \theta < 1$. Then

$$\sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))dx + O(1). \tag{2.36}$$

<u>Proof of Lemma 2.5</u>. Taking $\eta = 1 - \theta$ in Lemma 2.4 it is seen that the sum on the right-hand side of (2.33) reduces to the term m = 0, or it is empty

in case $f'(x) \geq \eta$ or $f'(x) < -\eta$, when

$$\int_a^b e(f(x))dx \ll 1$$

by Lemma 2.1, and (2.36) follows.

Lemma 2.6. Let f(x) be a real function for $a \leq x \leq b$ and let $H > 0$. Then

$$(2.37)\quad \sum_{a<n\leq b} e(f(n)) \ll (b-a)H^{-1/2} + H + \left\{(b-a)H^{-1}\sum_{h=1}^{H-1}\left|\sum_{a<n\leq b-h} e(f(n+h)-f(n))\right|\right\}^{1/2}$$

Proof of Lemma 2.6. We may suppose that a and b are integers and $2 \leq H < b-a$, since trivially $\sum_{a<n\leq b} e(f(n)) \ll b-a$ and $b-a \ll (b-a)H^{-1/2}$ for $H < 2$, while for $H \geq b-a$ the left-hand side of (2.37) is trivially majorized by H. Also we may suppose that H is an integer, since the right-hand side of (2.37) remains unchanged in magnitude if H is replaced by the integer nearest to it. Therefore the proof reduces to showing that

$$(2.38)\quad \sum_{a<n\leq b} e(f(n)) \ll (b-a)H^{-1/2} + \left\{(b-a)H^{-1}\sum_{h=1}^{H-1}\left|\sum_{a<n\leq b-h} e(f(n+h)-f(n))\right|\right\}^{1/2},$$

where H is an integer ≥ 2, $a < b$ are integers and $H < b-a$.

Observe that

$$(2.39)\qquad H\sum_{a<n\leq b} e(f(n)) = \sum_{m=1}^{H}\ \sum_{n=a-m+1}^{b-m} e(f(m+n)),$$

and define $f(k) = 0$ if k is an integer such that $k \leq a$ or $k > b$. Then writing $S = \sum_{a<n\leq b} e(f(n))$ and inverting the order of summation in (2.39) we obtain

$$(2.40)\qquad HS = \sum_{n=a+1-H}^{b-1}\ \sum_{m=1}^{H} e(f(m+n)),$$

so that n takes at most $b-a+H \leq 2(b-a)$ values. Applying the Cauchy-Schwarz inequality we have

$$(2.41)\qquad H^2S^2 \leq 2(b-a)\sum_{n=a+1-H}^{b-1}\left|\sum_{m=1}^{H} e(f(m+n))\right|^2.$$

Squaring out the modulus in (2.41) we obtain

$$(2.42)\quad \sum_{n=a+1-H}^{b-1}\left|\sum_{m=1}^{H} e(f(m+n))\right|^2 \leq 2(b-a)H + 2\left|\sum_{n=a+1-H}^{b-1}\ \sum_{1\leq r<s\leq H} e(f(n+s)-f(n+r))\right|.$$

In the last sum above for a fixed k,h such that $1 \le h \le H - 1$, $a < k \le b - h$ we have $f(n+s) - f(n+r) = f(k+h) - f(k)$ exactly $H - h$ times: for $r = 1,2,\dots,H-h, s=r+h, n=k-r$, and so the modulus of the double sum in (2.42) does not exceed

$$(2.43) \qquad \left|\sum_{h=1}^{H-1}(H-h)\sum_{a<k\le b-h} e(f(k+h)-f(k))\right| \le H\sum_{h=1}^{H-1}\left|\sum_{a<n\le b-h} e(f(n+h)-f(n))\right|,$$

and thus (2.38) follows easily from (2.40)-(2.43).

Finally we need a lemma which transforms an exponential sum into another exponential sum (plus error terms), and this new exponential sum is in many cases easier to estimate. This is

<u>Lemma 2.7</u>. Suppose that $f(x) \in C^4[a,b]$, $f'(x)$ is monotonically decreasing in $[a,b]$, $f'(b) = \alpha$, $f'(a) = \beta$. If x_ν is defined by $f'(x_\nu) = \nu$, ($\alpha < \nu \le \beta$ and ν is an integer) and

$$m_2 \asymp |f''(x)|,\ f^{(3)}(x) \ll m_3,\ f^{(4)}(x) \ll m_4,\ (m_3^2 = m_2 m_4),$$

then

$$(2.44) \qquad \sum_{a<n\le b} e(f(n)) = e(-1/8)\sum_{\alpha<\nu\le\beta}|f''(x_\nu)|^{-1/2}e(f(x_\nu) - \nu x_\nu) + O(m_2^{-1/2}) +$$

$$+ O((b-a)m_3^{1/3}) + O(\log((b-a)m_2 + 2)).$$

<u>Proof of Lemma 2.7</u>. We use Lemma 2.4, noting that by the mean value theorem

$$(2.45) \qquad \beta - \alpha \ll (b-a)m_2.$$

By Lemma 2.2 the limits of summation coming from Lemma 2.4 may be replaced by $\alpha + 1$ and $\beta - 1$ with an error $\ll m_2^{-1/2}$. An application of Theorem 2.1 gives then

$$\sum_{\alpha+1<\nu<\beta-1}\int_a^b e(f(x)-\nu x)\,dx = e(-1/8)\sum_{\alpha+1<\nu<\beta-1}|f''(x_\nu)|^{-1/2}e(f(x_\nu) - \nu x_\nu) +$$

$$+ O\Big(\sum_{\alpha+1<\nu<\beta-1} m_2^{-1}m_3^{1/3}\Big) + O\Big(\sum_{\alpha+1<\nu<\beta-1}((\nu-\alpha)^{-1} + (\beta-\nu)^{-1})\Big).$$

In view of (2.45) the first O-term above is $O((b-a)m_3^{1/3})$, and the second is $O(\log(\beta - \alpha + 2)) = O(\log((b-a)m_2 + 2)$, which ends the proof of (2.44), since again by Lemma 2.2 the limits of summation $(\alpha + 1, \beta - 1)$ may be changed to

(α,β) with an error $\ll m_2^{-1/2}$. It may be remarked that if we use (2.15) instead of (2.6) (with the appropriate hypotheses on f, of course) then we obtain (2.44) with the error term $O((b-a)m_3^{1/3})$ replaced by $O((b-a)m_2^{1/5}m_3^{1/5})$. Also if f' is monotonically increasing in $[a,b]$ and the other hypotheses of Lemma 2.1 are the same, but $f'(a) = \alpha, f'(b) = \beta$, then (2.44) remains true with e(-1/8) replaced by e(1/8).

§3. The theory of exponent pairs

We have now at our disposal two results, namely Lemma 2.6 and Lemma 2.7, which enable us to transform a given exponential sum into other exponential sums plus some (usually manageable) error terms. Lemma 2.6 requires practically no conditions on f, while Lemma 2.7 is much more restrictive and contains error terms. However the conditions imposed on the derivatives of f in Lemma 2.7 allow us for a large class of functions f (which occur in many important applications) to combine Lemma 2.6 and Lemma 2.7 successfully several times and to obtain good upper bounds for the modulus of

(2.46) $$S = \sum_{B<n\leq B+h} e(f(n)), \quad (B \geq 1,\ 0 < h \leq B)$$

provided that f satisfied certain conditions. The results of §2 suggest that the estimation of the exponential sum S certainly depends on the number of summands, which is $\leq B$, and on the size of the first derivative of f. Therefore we shall suppose that

(2.47) $$A \ll |f'(x)| \ll A, \quad (A > 1/2)$$

when $B \leq x \leq 2B$, and seek an upper bound for $|S|$ of the form

(2.48) $$S \ll A^p B^q.$$

The pair of non-negative real numbers (p,q) will be called an <u>exponent pair</u> if (2.47), (2.48) hold and

(2.49) $$0 \leq p \leq 1/2 \leq q \leq 1.$$

Two remarks may be immediately made here: firstly that $(p,q) = (0,1)$ is trivially an exponent pair, and secondly that exponent pairs obviously form a convex set. This is to be understood in the following sense: if (p_1,q_1) and (p_2,q_2) are arbitrary exponent pairs and $0 \leq t \leq 1$ is arbitrary, then

(2.50) $$S = S^t S^{1-t} \ll A^{p_1 t+(1-t)p_2} B^{q_1 t+(1-t)q_2},$$

which implies that

(2.51) $$(p_1 t + (1-t)p_2, q_1 t + (1-t)q_2), \quad (0 \le t \le 1)$$

is also an exponent pair. The above definition of exponent pairs is too general, and to obtain exponent pairs of practical value (via Lemma 2.7) we shall suppose that besides (2.47) $f(x) \in C^r[B,2B]$ for some $r \ge 5$, and moreover that the derivatives of f for $B \le x \le 2B$ satisfy

(2.52) $$AB^{1-r} \ll |f^{(r)}(x)| \ll AB^{1-r}, \quad (r = 1,2,\ldots)$$

wherethe $\ll$-constants in (2.52) depend on r alone.

We may consider only the case $f'(x) > 0$ for $B \le x \le 2B$, since otherwise we may consider $\overline{S}$ instead of S with the effect that f is replaced by -f and the sign of f' is thus changed. To obtain the first non-trivial exponent pair we apply Lemma 2.4 to S, estimating each integral as $\ll m_2^{-1/2} \ll (A/B)^{-1/2}$ by Lemma 2.2. This yields

(2.53) $$S \ll (\beta - \alpha)A^{-1/2}B^{1/2} + A^{-1/2}B^{1/2} + \log(2 + A) \ll (AB)^{1/2},$$

since $\beta - \alpha = f'(a) - f'(b) \ll A$ and $A \gg 1$. Therefore it follows that $(p,q) = (1/2,1/2)$ is an exponent pair, where (2.52) was used with $r = 1$ and $r = 2$ only. Thus we have so far $(0,1),(1/2,1/2)$ as exponent pairs, plus exponent pairs which may be formed from these two and convexity (in the sense of (2.51)). Denote this set of exponent pairs by E_1. New exponent pairs which do not belong to E_1 may be obtained by using exponent pairs from E_1, convexity and the following

<u>Lemma 2.8</u>. If (p,q) ia an exponent pair, then so is also

$$(k,l) = (p/(2p+2), 1/2 + q/(2p+2)).$$

<u>Proof of Lemma 2.8</u>. First note that $0 \le k \le 1/2 \le l \le 1$, since $0 \le p \le 1/2 \le q \le 1$ by hypothesis. An application of Lemma 2.6 gives

(2.54) $$S^2 \ll B^2H^{-1} + H^2 + BH^{-1}\sum_{j=1}^{H-1}\Big|\sum_{B<n\le B+h-j} e(f(n+j)-f(n))\Big|,$$

where $H > 0$ will be suitably chosen. For a fixed j write

(2.55) $$g(n) = f(n+j) - f(n),$$

and note that $g^{(r)}(x) = f^{(r)}(x+j) - f^{(r)}(x) = jf^{(r+1)}(x+\theta j), \quad (|\theta| \le 1)$

so that by (2.52) for $B \le x \le B + h - j$

$$jAB^{-r} \ll |g^{(r)}(x)| \ll jAB^{-r}. \quad (r = 1,2,\ldots) \tag{2.56}$$

Now we may suppose that $A > B^{1/2}$, since for $A \le B^{1/2}$ we use the fact that $q \ge 1/2$ and $(1/2,1/2)$ is an exponent pair to obtain

$$S \ll A^{1/2}B^{1/2} \ll A^{1/2}B^{1/2+q/(2p+2)}B^{-q/(2p+2)} \ll A^{(1+p-2q)/(2p+2)}B^{l} \ll A^{k}B^{l},$$

where (k,l) is as in the formulation of the lemma.

The condition $A > B^{1/2}$ is needed in the case when for some $c > 0$ we have $j < cB/A$, so that by (2.56) $|g'| < 1/2$, $g'' \asymp jAB^{-2} \ll 1$. Then by (2.36) and Lemma 2.2

$$BH^{-1}\sum_{j<cB/A}\Big|\sum_{n} g(n)\Big| \ll BH^{-1}A^{-1/2}B\sum_{j<cB/A} j^{-1/2} \ll B^{2}H^{-1},$$

since $A > B^{1/2}$. For the remaining j's in (2.54) we use the already existing (p,q) and (2.56) to obtain

$$\sum_{B<n\le B+h-j} e(f(n+j)-f(n)) \ll (jAB^{-1})^{p}B^{q}.$$

Hence by (2.54)

$$S^{2} \ll B^{2}H^{-1} + A^{p}B^{1-p+q}H^{p} + H^{2}, \tag{2.57}$$

and the choice $H^{p+1} = B^{1+p-q}A^{-p}$ finally reduces (2.57) to

$$S^{2} \ll A^{p/(1+p)}B^{(1+p+q)/(1+p)} + B^{2(1+p-q)/(1+p)}A^{-2p/(1+p)}. \tag{2.58}$$

Since $0 \le p \le 1/2 \le q \le 1$ we have

$(1 + p + q)/(1 + p) = 1 + q/(p + 1) \ge 4/3 \ge$

$2(1 - q/(1 + p)) = 2(1 + p - q)/(1 + p),$

so that in view of $A > 1/2$ the second term in (2.58) does not exceed the first and Lemma 2.8 follows.

Now we denote by E_2 the set of exponent pairs obtainable from E_1, convexity and repeated application of Lemma 2.8, which always produces a new exponent pair from a given (p,q). The proof of Lemma 2.8 shows that for the construction of E_2 we needed (2.52) with $r \le 3$. The set E_2 does not exhaust our possibilities for constructing exponent pairs, and for what follows it will be useful to note that for $(p,q) \in E_2$ we have

(2.59) $$p + 2q \geq 3/2.$$

This is trivial if $(p,q) \in E_1$, and moreover convexity obviously preserves (2.59). With $k = p/(2p + 2)$, $l = 1/2 + q/(2p + 2)$ it is however readily checked that $k + 2l \geq 3/2$ since $q \geq 1/2$.

Finally the last possibility for constructing exponent pairs is furnished by

Lemma 2.9. If (p,q) is an exponent pair for which (2.59) holds, then $(k,l) = (q - 1/2, p + 1/2)$ is also an exponent pair.

Proof of Lemma 2.9. The condition $0 \leq k \leq 1/2 \leq l \leq 1$ is trivial in view of (2.49). We shall apply Lemma 2.7 with $a = B, b = B + h$, $m_2 = AB^{-1}$, $m_3 = AB^{-2}$, $m_4 = AB^{-3}$ so that $m_3^2 = m_2 m_4$ holds, and we may suppose that $f''(x) < 0$; the case $f''(x) > 0$ is discussed at the end of Lemma 2.7 and will lead to the same final estimate. We have then

(2.60) $$S = e(-1/8) \sum_{\alpha < \nu \leq \beta} |f''(x_\nu)|^{-1/2} e(f(x_\nu) - \nu x_\nu) + O(A^{-1/2}B^{1/2}) + O(\log(A+2)) + O((AB)^{1/3}),$$

and the main task is to estimate

(2.61) $$S_1 = \sum_{\alpha < \nu \leq \beta} |f''(x_\nu)|^{-1/2} e(g(\nu)), g(\nu) = f(x_\nu) - \nu x_\nu.$$

If we set $f'(x) = y$ and denote its inverse function by $x = h(y)$, then $g(y) = f(h(y)) - yh(y)$, which gives

$$g'(y) = f'(h(y))h'(y) - h(y) - yh'(y) = f'(x)h'(y) - h(y) - yh'(y) = -h(y),$$

$$g''(y) = -h'(y) = -1/f''(x) = -1/f''(h(y))$$

if one uses $f'(h(y)) = y$, and

$$g^{(3)}(y) = f^{(3)}(h(y))(f''(h(y)))^{-3},$$

and in general $g^{(r)}(y)$ is found from $g''(y)f''(h(y)) = -1$ by applying Leibniz's rule for the r-th derivative of a product. We have $h(y) \asymp B$ and therefore

(2.62) $$BA^{1-r} \ll |g^{(r)}(x)| \ll BA^{1-r}, \quad (r \leq 3)$$

and by induction it may be seen that the upper bound in (2.62) holds also for $r \geq 4$. Removing f'' by partial summation it further follows that S_1 in (2.61) is of the same type as S, only A and B are interchanged. Hence

(2.63) $$S_1 \ll A^{-1/2}B^{1/2}B^pA^q,$$

(2.64) $$S \ll A^kB^l + (AB)^{1/3}, \ k = q - 1/2, l = p + 1/2,$$

and the proof will be finished if it can be shown that $(AB)^{1/3} \leq A^kB^l$. Since $l \geq 1/2$ this is obvious if $k \geq 1/3$. If $k < 1/3$, then for $B \geq A^{(1-3k)/(3l-1)}$ we have

$$(AB)^{1/3} = A^kB^{1/3}A^{1/3-k} \leq A^kB^{1/3}B^{(3l-1)/3} = A^kB^l.$$

For $B < A^{(1-3k)/(3l-1)}$ we have $2k + l \geq 1$ from $p + 2q \geq 3/2$, which gives

$$S \leq B = B^lB^{1-l} < B^lA^{(1-3k)(1-l)/(3l-1)} \leq A^kB^l,$$

ending the proof of Lemma 2.9, where (2.52) was needed for $r \leq 3$ and the upper bound of (2.52) for $r = 4$.

In view of the preceding discussion we formalize now the concept of exponent pairs even more by introducing E, the set of exponent pairs,as the set obtainable from E_2, convexity and Lemma 2.9 applied a finite number of times. Nearly sixty years of research have not been able to produce any other exponent pairs,i.e. any besides those of E, where f is a real-valued function satisfying conditions (similar to) (2.47) and (2.52). Though in the formulation of Lemma 2.8 and Lemma 2.9 it was tacitly assumed that (p,q) belongs to E_1 and E_2 respectively, this is not necessarily true, as the proof of these lemmas clearly shows. It seems appropriate now to introduce three processes which will be denoted by A,B,C(t) (the letters A and B have no relation to (2.48) in this context), and which correspond to Lemma 2.8, Lemma 2.9 and convexity respectively. Therefore if (p,q) and (p_1,q_1) are exponent pairs, let

$$A(p,q) = (p/(2p+2), 1/2 + q/(2p+2)),$$

$$B(p,q) = (q - 1/2, p + 1/2),$$

$$C(t)(p,q)(p_1,q_1) = (pt + p_1(1-t), qt + q_1(1-t)). \quad (0 \leq t \leq 1)$$

Now we can reinterpret the theory of exponent pairs and state the following

<u>Proposition</u>. Let E denote the set of pairs of real numbers (p,q) such that $0 \leq p \leq 1/2 \leq q \leq 1$ and (p,q) is obtained by a finite number of applications of the processes A,B and C(t) defined above to (0,1), which is to be considered as

an element of E. Then E is the set of exponent pairs in the sense that (2.48) holds, provided that (2.52) holds with $r \geq 4$.

We end this chapter by giving several of the most commonly used exponent pairs: $(1/2,1/2) = B(0,1)$, $(1/6,2/3) = A(1/2,1/2)$, $(2/7,4/7) = BA(1/6,2/3)$, $(4/18,11/18) = BA(2/7,4/7)$, $(11/30,16/30) = BA^2(1/6,2/3)$, $(13/40,22/40) = BA^2(2/7,4/7)$, $(97/251,132/251) = BA^3(13/40,22/40)$, $(13/31,16/31) = BABA(11/30,16/30)$, $(5/24,15/24) = C(1/4)(1/6,2/3)(4/18,11/18)$, $(4/11,6/11) = C(12/33)(1/2,1/2)(2/7,4/7)$.

It may be remarked that in actual problems where the theory of exponent pairs is applied it often seems unclear how to choose (p,q) in an optimal way, i.e. to minimize a certain function $F(p,q)$. In the case of the general F this problem is difficult and to this day unsolved, but for $F(x,y) = x + y$ it has been solved by R.A. Rankin [1], who showed that if $\alpha = 0.329021358\ldots$, then $(p,q) = (\alpha/2+\varepsilon, 1/2+\alpha/2+\varepsilon)$ is an exponent pair for which (up to ε) $p + q$ is minimal for all (p,q) belonging to E. Graphically the exponent pairs just discussed may be arranged in a table as follows:

p	0	1/2	1/6	2/7	4/18	11/30	13/40	13/31	4/11	5/24	97/251	$\alpha/2+\varepsilon$
q	1	1/2	2/3	4/7	11/18	16/30	22/40	16/31	6/11	15/24	132/257	$1/2+\alpha/2+\varepsilon$

NOTES

The results presented in this chapter have their counterparts in Chapter 4 and Chapter 5 of Titchmarsh [8], but the material given here is more extensive and the results sharper. In particular, Titchmarsh does not present the theory of exponent pairs, but stops at what is essentially Lemma 2.8 applied several times to the exponent pair (1/2,1/2); this is Titchmarsh's Theorem 5.13.

The theory of exponent pairs, exponential sums and integrals has been founded by J.G. van der Corput [1], [2] in the 1920's as one of the deepest theories of analytic number theory ever made. Van der Corput [2] contains the estimate $\Delta(x) \ll x^{33/100+\varepsilon}$ (where $\Delta(x)$ is the error term in the clasical divisor problem for which the reader is referred to Chapter 10), which was a very important improvement of the previous exponent 1/3, due to G.F. Voronoï [1]. The exponent 1/3 appeared also in the circle problem (i.e. determining Θ such that $\sum_{m^2+n^2\le x} 1 - \pi x \ll x^{\Theta+\varepsilon}$;

see also Chapter 10 for a more extensive discussion of the circle problem), and until van der Corput's results appeared many competent mathematicians believed that the exponent 1/3 was the natural limit of the existing methods, if not nearly the true order of magnitude. Van der Corput's research opened a path in analytic number theory which leads to good bounds in many important problems, and forms the basis for more advanced methods.

The original form of van der Corput's theory was rather complicated, and his definition of exponent pairs involves a condition comparable to (2.52): (p,q) is an exponent pair if (2.49) holds and if, corresponding to every $s > 0$, there exist two numbers r and c depending only on s ($r \geq 4$ is an integer and $0 < c < 1/2$) such that

$$\sum_{a<n\leq b} e(f(n)) \ll z^p a^q \tag{2.65}$$

holds with the $\ll$-constant depending only on s and u, where $u > 0, 1 \leq a < b < au$, $y > 0$, $z = ya^{-s} > 1$, $f(n)$ is any real function with differentiable coefficients in the interval $a \leq n \leq b$ (a,b integers) of the first r orders and for $a \leq n \leq b$, $0 \leq j \leq r-1$

$$|f^{(j+1)}(n) - (-1)^j ys(s+1)\dots(s+j-1)n^{-s-j}| < cys(s+1)\dots(s+j-1)j^{-s-j}. \tag{2.66}$$

It may be noted that z is effectively $f'(a)$, so that (2.65) is in fact the same as (2.48), and the only difference is between (2.52) and (2.66) which express the same type of inequalities for derivatives of f, only (2.52) is simpler to verify and thus the definition of the exponent pair made in the text is more practical, though in most common applications (e.g. the divisor problem, the order of $\zeta(1/2+it)$ etc.) the function f is easily seen to satisfy both definitions of exponent pairs.

Lemmas 2.1, 2.2, 2.4 and 2.5 are to be found also in Chapter 5 of Titchmarsh [8], and also a variant of Lemma 2.6 is given by Titchmarsh [8] as well as the proof of (2.15). Lemma 2.4, and its Corollary Lemma 2.5, may be viewed as a consequence of the Poisson summation formula (1.23).

Great simplifications in van der Corput's theory were introduced by E. Phillips [1], whose proofs of Lemma 2.8 and Lemma 2.9 are essentially given here, and the theory of exponent pairs was brought to a readily applicable form in the Proposition at the end of the chapter, where several commonly used exponent pairs were constructed.

Theorem 2.1 is also due to E. Phillips [1], while Theorem 2.2 is due to F.V. Atkinson [3] and will be used in Chapter 4 and Chapter 6 for transformations of certain Dirichlet polynomials (finite Dirichlet series) via the Voronoï summation formula (see Chapter 3) and finally Theorem 2.2 will be used in Chapter 11 for the derivation of Atkinson's formula [3] for $E(T)$.

Lemma 2.3, concerning integrals with no saddle points, is due to M. Jutila [6]. This result will serve as a useful device in Chapter 6 for the truncation of series when Voronoï's summation formula is applied.

The main step in the proof of Lemma 2.6 is the inequality (2.38), due originally to H. Weyl. This inequality is of a general nature and rests on a judicious use of the Cauchy-Schwarz inequality. In fact the sum appearing on the left-hand side of (2.43) is a double exponential sum (since $H - h$ can be removed by partial summation), so that Lemma 2.6 in fact transforms an (ordinary) exponential sum into a double exponential sum with the flexibility that H may be chosen suitably to minimize the estimates. Thus in (2.43) one may see the genesis of two- and multi-dimensional methods in the estimation of exponential sums. The method of two-dimensional sums was developed in the 1930's by E.C. Titchmarsh [1], [2], [3], [6] where he obtained several improvements of exponents in the classical problems such as the order of $\zeta(1/2+it)$ and the circle problem. One of his early results, which may be regarded as the two-dimensional analogue of Lemma 2.1, is as follows: Let D be a finite region bounded by $O(1)$ continuous monotonic arcs which is included in the square $|x| \le R$, $|y| \le R$ for some $R \ge 2$. Let further $f_{xx}(x,y) > 0, f_{yy}(x,y) < 0$ (or $f_{xx} < 0, f_{yy} > 0$) and $f_{xy}(x,y) \ge b > 0$ throughout D. Then

$$\iint_D e(f(x,y))dxdy \ll b^{-1}(\log R + |\log b|).$$

Later refinements of two-dimensional methods were effected by many mathematicians, including S.-H. Min [1], H.-E. Richert [1] and W. Haneke [1]. The best methods at present are those of G. Kolesnik [1] - [6], as already mentioned in §1. These advanced techniques, which do not seem to have appeared in book form yet, are very complicated and are based on works of previous authors, so that anyone who wants to get acquainted with them must know some of the theory already. An attempt to create the theory of n-dimensional exponent pairs has been made by B.R. Srinivasan [1], [2], [3]. As is the case with the theory of exponent pairs which is presented here, Srinivasan's theory is readily applicable too, but in each specific problem the more advanced methoda of the aforementioned papers of G. Kolesnik will lead to a better result: witness the recent improvement of 105/407 = 0.257985... in the problem of non-isomorphic abelian groups (Srinivasan [3]) to 97/381 = 0.254593... by G. Kolesnik [7]. As mentioned at the beginning of this chapter, all existing multi-dimensional methods do not improve very much the results obtainable by the method of exponent pairs. A discussion of the results obtainable by the method of exponent pairs was made by R.A. Rankin [1], where he showed that the best exponent (up to ε) that this method (at present) allows in the divisor problem is 0.3290213568..., while the best known exponent (due to G. Kolesnik [6]) is 35/108 = 0.32407407407... . This is one of the reasons why our discussion of exponential sums was limited to the method of exponent pairs, which though not optimal is sufficiently good for many applications.

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TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 3

THE VORONOÏ SUMMATION FORMULA

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THE VORONOÏ SUMMATION FORMULA

§1. Introduction

At the beginning of this century G.F. Voronoï [1] proved two remarkable formulas concerning the error term in the divisor problem. Besides giving an explicit expression for the error term $\Delta(x)$, Voronoï derived a general summation formula for sums involving the divisor function $d(n)$. The formulas of Voronoï express finite arithmetical sums by infinite series containing the Bessel functions, and they are

$$(3.1)\quad \Delta(x) = {\sum_{n\le x}}' d(n) - x(\log x + 2\gamma - 1) - 1/4 =$$

$$= -2\pi^{-1}x^{1/2}\sum_{n=1}^{\infty} d(n)n^{-1/2}\left(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})\right),$$

and

$$(3.2)\quad {\sum_{a\le n\le b}}' d(n)f(n) = \int_a^b (\log x + 2\gamma)f(x)\,dx + \sum_{n=1}^{\infty} d(n)\int_a^b f(x)\alpha(nx)\,dx,$$

where

$$(3.3)\quad \alpha(x) = 4K_0(4\pi x^{1/2}) - 2\pi Y_0(4\pi x^{1/2}).$$

Here $0 < a < b < \infty$, $f(x) \in C^2[a,b]$, $\sum'$ means that if a or b are integers then $\frac{1}{2}f(a)$ or $\frac{1}{2}f(b)$ is to be counted instead of $f(a)$ and $f(b)$ respectively in (3.2). The series in (3.1) and (3.2) are boundedly convergent when x or a and b lie in a fixed closed subinterval of $(0,\infty)$. The functions K_0, K_1, Y_0, Y_1 are the familiar Bessel functions with power series expansions

$$(3.4)\quad K_0(z) = -(\log(z/2) + \gamma)I_0(z) + \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2}(1 + 1/2) + \frac{z^6}{2^2 4^2 6^2}(1 + 1/2 + 1/3) + \ldots$$

$$(3.5)\quad Y_0(z) = \frac{2}{\pi}(\log(z/2) + \gamma)J_0(z) + \frac{2}{\pi}\left(\frac{z^2}{2^2} - \frac{z^4}{2^2 4^2}(1 + 1/2) + \frac{z^6}{2^2 4^2 6^2}(1 + 1/2 + 1/3) - \ldots\right.$$

where

$$I_0(z) = \sum_{m=0}^{\infty}(z/2)^{2m}/(m!)^2,\quad J_0(z) = \sum_{m=0}^{\infty}(-1)^m(z/2)^{2m}/(m!)^2,$$

$$(3.6)\qquad K_1(z) = \sum_{m=0}^{\infty}\frac{(z/2)^{2m+1}}{m!\,(m+1)!}(\log(z/2) - \tfrac{1}{2}\psi(m+1) - \tfrac{1}{2}\psi(m+2)),$$

$$(3.7)\qquad Y_1(z) = \pi^{-1}\sum_{m=0}^{\infty}(-1)^m\frac{(z/2)^{2m+1}}{m!\,(m+1)!}(2\log(z/2) - \psi(m+1) - \psi(m+2)),$$

where γ is Euler's constant and in this chapter only

$$(3.8)\qquad \psi(z) = \Gamma'(z)/\Gamma(z).$$

From $\Gamma(1+z) = z\Gamma(z)$ and $\psi(1) = -\gamma$ (see (1.30)) it is seen that for $n \geq 2$ we have

$$\psi(n) = 1 + 1/2 + \ldots + 1/(n-1) - \gamma .$$

The above functions are usually called the modified Bessel functions, arising from

$$(3.9)\qquad J_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{p+2k}}{k!\,\Gamma(p+k+1)},\ I_p(z) = e(-p/4)J_p(iz) = \sum_{k=0}^{\infty}\frac{(z/2)^{p+2k}}{k!\,\Gamma(p+k+1)},$$

where the parameter p is a fixed real number. In fact for any integer n one may define

$$(3.10)\quad K_n(z) = \frac{(-1)^n}{2}\left\{\frac{\partial I_{-p}(z)}{\partial p} - \frac{\partial I_p(z)}{\partial p}\right\}_{p=n},\ Y_n(z) = \frac{1}{\pi}\left\{\frac{\partial J_p(z)}{\partial p} - (-1)^n\frac{\partial J_{-p}(z)}{\partial p}\right\}_{p=n},$$

and then from (3.9) deduce (3.4)-(3.7) and also

$$(3.11)\qquad \frac{d}{dx}(xK_1(x)) = -xK_0(x),\ \frac{d}{dx}(xY_1(x)) = xY_0(x).$$

The practical value of the formulas (3.1) and (3.2) lies in the fact that the Bessel functions appearing in them admit sharp asymptotic approximations involving elementary functions, which are valid for $|z|$ large and $|\arg z| < \pi$. Defining for $p \geq 0$ real and $m \geq 0$ an integer

$$(p,m) = \frac{\Gamma(p+m+1/2)}{m!\,\Gamma(p-m+1/2)} = \frac{(4p^2-1^2)(4p^2-3^2)\ldots(4p^2-(2m-1)^2)}{2^{2m}\cdot m!},$$

we have for $|z|$ large and $|\arg z| < \pi$

$$(3.12)\qquad K_p(z) \sim (\pi/2z)^{1/2}e^{-z}\sum_{m=0}^{\infty}(p,m)(2z)^{-m},$$

(3.13) $$Y_p(z) \sim (2/\pi z)^{1/2}\Big\{\sin(z-\tfrac{1}{2}\pi p-\tfrac{\pi}{4})\sum_{m=0}^{\infty}(-1)^m(p,2m)(2z)^{-2m} +$$

$$+ \cos(z-\tfrac{1}{2}\pi p-\tfrac{\pi}{4})\sum_{m=0}^{\infty}(-1)^m(p,2m+1)(2z)^{-2m-1}\Big\},$$

(3.14) $$J_p(z) \sim (2/\pi z)^{1/2}\Big\{\cos(z-\tfrac{1}{2}\pi p-\tfrac{\pi}{4})\sum_{m=0}^{\infty}(-1)^m(p,2m)(2z)^{-2m} -$$

$$- \sin(z-\tfrac{1}{2}\pi p-\tfrac{\pi}{4})\sum_{m=0}^{\infty}(-1)^m(p,2m+1)(2z)^{-2m-1}\Big\}.$$

The symbol $\sim$ means here that in (3.12)-(3.14) equality holds if in the sums over m we stop at any finite term and multiply it by $1 + O(|z|^{-1})$, With the above formulas we obtain from (3.3)

(3.15) $$\alpha(nx) = -2^{1/2}x^{-1/4}n^{-1/4}\Big\{\sin(4\pi\sqrt{nx}-\pi/4) - (32\pi)^{-1}(nx)^{-1/2}\cos(4\pi\sqrt{nx}-\pi/4)\Big\} + O(n^{-5/4}x^{-5/4}),$$

which is sufficiently sharp for most applications of the summation formula (3.2).

§2. The truncated Voronoi formula

There are no simple proofs of (3.1) and (3.2), where delicate questions of convergence are involved. The most difficult case in (3.1) is when x is an integer, but in most applications the distinction whether x is an integer or not is not important, since $d(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$. In practice it is useful to have a truncated form of (3.1), and Chapter 12 of Titchmarsh's book [8] contains a proof of

(3.16) $$\Delta(x) = -2\pi^{-1}x^{1/2}\sum_{n\le N}d(n)n^{-1/2}\big(K_1(4\pi\sqrt{nx})+\tfrac{\pi}{2}Y_1(4\pi\sqrt{nx})\big) + O(x^{\varepsilon}) + O(x^{1/2+\varepsilon}N^{-1/2})$$

which in view of (3.12) and (3.13) may be replaced by the simpler expression

(3.17) $$\Delta(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le N}d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4) + O(x^{\varepsilon}) + O(x^{1/2+\varepsilon}N^{-1/2}).$$

Here N is a (sufficiently large) parameter which may be suitably chosen; the choice $N = x^{1/3}$ implies immediately $\Delta(x) \ll x^{1/3+\varepsilon}$, while letting $N \to \infty$ in (3.16) we obtain (3.1) in a weaker form with the error term $O(x^{\varepsilon})$ present. The proof of (3.16), as presented by Titchmarsh [8], starts from Perron's inversion formula (1.10) which gives

$$\sum_{n\le x}{}' d(n) = (2\pi i)^{-1}\int_{c-iT}^{c+iT} \zeta^2(s)x^s s^{-1}\,ds + O(x^c T(c-1)^{-2}) + O(x^{1+\epsilon}T^{-1}),$$

where $c = 1 + 1/\log x$, $T^2/(4\pi^2 x) = N + 1/2$, and N is an integer. Here N is the same parameter which appears in (3.16) and (3.17), since in those formulas it is irrelevant whether N is an integer or not if $N \ll x^A$ for some fixed $A > 0$, which we henceforth assume. The contour in the above integral is replaced by the contour joining the points $c \pm iT$, $-a \pm iT$ $(a > 0)$. Allowing for the poles at $s = 0$ and $s = 1$, we obtain by the residue theorem

$$(3.18)\qquad \Delta(x) = (2\pi i)^{-1}\sum_{n=1}^{\infty} d(n)\int_{-a-iT}^{-a+iT} \chi^2(s)n^{s-1}x^s s^{-1}\,ds + O(x^{\epsilon}) + O(T^{2a}x^{-2a}) + O(x^{1+\epsilon}T^{-1}),$$

where the functional equation (4.3) was used, and $\zeta^2(1-s)$ was replaced by the absolutely convergent series which may be integrated term by term, so that (3.18) follows on estimating the integrals over the horizontal segments. Using Lemma 2.1 and the asymptotic formula (4.4) for $\chi(s)$ it is seen that the contribution of $\sum_{n>N}$ in (3.18) is contained in the error terms, and writing

$$\int_{-a-iT}^{-a+iT} \chi^2(s)n^{s-1}x^s s^{-1}\,ds = \int_{-i\infty}^{i\infty} - \left(\int_{iT}^{i\infty} + \int_{-i\infty}^{-iT} + \int_{-iT}^{-a-iT} + \int_{-a+iT}^{iT}\right),$$

we obtain, after estimating the integrals in the bracket above either trivially or by Lemma 2.1,

$$(3.19)\qquad \Delta(x) = (2\pi i)^{-1}\sum_{n\le N} d(n)n^{-1}\int_{-i\infty}^{i\infty} 2^{2s}\pi^{2s-2}\sin^2\pi s/2\cdot\Gamma^2(1-s)(nx)^s s^{-1}\,ds \; +$$

$$+ \; O(x^{\epsilon}) + O(x^{1/2+\epsilon}N^{-1/2}),$$

where $a = \epsilon$, $1 \ll N \ll x^A$, and where the expression (4.3) for $\chi(s)$ was used. The above transformations of $\Delta(x)$ were necessary, since the change of variable $s = 1 - w$ in (3.19) shows that (3.16) follows from

$$-(\pi^2 n)^{-1}\int_{1-i\infty}^{1+i\infty} \cos^2\pi w/2\cdot\Gamma(w)\Gamma(w-1)(2\pi\sqrt{nx})^{2-2w}\,dw =$$

$$-4i(x/n)^{1/2}(K_1(4\pi\sqrt{nx}) + \tfrac{\pi}{2}Y_1(4\pi\sqrt{nx}),$$

or by writing $2\pi\sqrt{nx} = X$ we have to show

$$(3.20) \qquad (2\pi i)^{-1}\int_{1-i\infty}^{1+i\infty} \cos^2 \pi w/2 \cdot \Gamma(w)\Gamma(w-1) X^{2-2w} dw = X(K_1(2X) + \tfrac{\pi}{2} Y_1(2X)).$$

To see that (3.20) holds note that

$$(3.21) \qquad f(x) = x^{-1}Y_1(x),\ F(s) = -2^{s-2}\pi^{-1}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s}{2}-1)\cos\pi(\tfrac{s}{2}-1), \quad (2 < \delta < 5/2)$$

$$(3.22) \qquad f(x) = x^{-1}K_1(x),\ F(s) = 2^{s-3}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s}{2}-1) \qquad (\delta > 2)$$

are respective Mellin transforms in the sense of (1.1) and (1.3), and thus by (1.3)

$$\tfrac{\pi}{2}X^{-1}Y_1(X) + X^{-1}K_1(X) = (2\pi i)^{-1}\int_{1-i\infty}^{1+i\infty} 2^{s-3}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s}{2}-1)(\cos\tfrac{\pi s}{2} + 1)X^{-s} ds =$$

$$2(2\pi i)^{-1}\int_{1-i\infty}^{1+i\infty} 2^{2w-3}\Gamma(w)\Gamma(w-1)(\cos\pi w + 1)X^{-2w} dw = (2\pi i)^{-1}\int_{1-i\infty}^{1+i\infty} 2^{2w-1}\cos^2\pi w/2 \cdot \Gamma(w)\Gamma(w-1)X^{-2w} dw.$$

Replacing X by 2X we obtain

$$\tfrac{\pi}{2}X^{-1}Y_1(2X) + X^{-1}K_1(2X) = (2\pi i)^{-1}\int_{1-i\infty}^{1+i\infty} \cos^2\pi w/2 \cdot \Gamma(w)\Gamma(w-1)X^{-2w} dw,$$

and finally multiplication by X^2 proves (3.20) and therefore (3.16) too. A similar formula may be derived by this method for $\Delta_k(x)$, the error term in the asymptotic formula for $\sum_{n\le x} d_k(n)$ (see Chapter 10), and for $k \ge 2$ fixed one obtains

$$(3.23) \qquad \Delta_k(x) \ll x^{(k-1)/2k}\Big|\sum_{n\le N} d_k(n)n^{-(k+1)/2k}e(k(xn)^{1/k})\Big| + x^{\epsilon} + x^{(k-1+\epsilon)/k}N^{-1/k}.$$

§3. The weighted Voronoi formula

An effective way to prove both (3.1) and (3.2) is to consider the weighted sum

$$(3.24) \qquad D_{q-1}(x) = \frac{x^{q-1}}{\Gamma(q)}\sum_{n\le x}(1 - n/x)^{q-1}d(n), \qquad (q \ge 1)$$

and find an asymptotic expansion of this sum (q is a fixed number, not necessarily an integer) in terms of some "generalized Bessel functions". This line of approach has been successfully used by several authors, and we shall follow here the work of A.L. Dixon and W.L. Ferrar [1]. By the inversion formula (1.14) we have

$$D_{q-1}(x) = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty} \zeta^2(s)\frac{x^{s+q-1}\Gamma(s)}{\Gamma(s+q)}ds,$$

which is the starting point for the evaluation of $D_{q-1}(x)$. For $q > 2$ and $0 < c < \min(1/2, q/2-1)$ it is seen by Stirling's formula (1.32) that the line of integration in the above integral may be replaced by the line $\operatorname{Re} s = -c$, and hence by the residue theorem

$$(3.25)\qquad D_{q-1}(x) = (2\pi i)^{-1}\int_{-c-i\infty}^{-c+i\infty}\zeta^2(s)\frac{x^{s+q-1}\Gamma(s)}{\Gamma(s+q)}ds + \frac{x^{q-1}}{4\Gamma(q)} + \frac{x^q}{\Gamma(q+1)}(\gamma + \log x - \Psi(1+q))$$

since the integrand has a simple pole at $s = 0$ and a double pole at $s = 1$. As in the proof of (3.16) we use now the functional equation for the zeta-function and replace $\zeta^2(1-s)$ by the absolutely convergent series which may be integrated termwise to give

$$(2\pi i)^{-1}\int_{-c-i\infty}^{-c+i\infty}\zeta^2(s)\frac{x^{s+q-1}\Gamma(s)}{\Gamma(s+q)}ds = (4^q\pi^{2q-2})^{-1}\sum_{n=1}^{\infty} d(n)n^{-q}(2\pi i)^{-1}\int_{-c-i\infty}^{-c+i\infty}\frac{(4\pi^2nx)^{s+q-1}}{\Gamma(s)\Gamma(s+q)\cos^2\pi s/2}ds,$$

since by Stirling's formula the integrand on the left-hand side above is absolutely convergent. For n and x fixed the second integral above is equal to minus $2\pi i$ times the sum of residues at its double poles $s = 2m+1$ $(m = 0,1,2,\ldots)$. To calculate these residues observe that $\cos z$ is an even function of z and for $z = s - (2m+1)$

$$\cos^2\pi s/2 = \cos^2(\pi z/2 + (2m+1)\tfrac{\pi}{2}) = \sin^2\pi z/2 = (1 + o(1))\frac{\pi^2 z^2}{4},$$

while the linear part of the expansion of

$$\frac{(4\pi^2nx)^{s+q-1}}{\Gamma(s)\Gamma(s+q)} = \frac{f(s)}{g(s)}$$

at $s = 2m+1$ is equal by Taylor's formula to

$$\left(\frac{f}{g}\right)'\Bigg|_{s=2m+1} = \frac{f}{g}\left(\frac{f'}{f} - \frac{g'}{g}\right)\Bigg|_{s=2m+1} =$$

$$\frac{(4\pi^2nx)^{2m+q}}{\Gamma(2m+1)\Gamma(2m+1+q)}(\log(4\pi^2nx) - \Psi(2m+1) - \Psi(2m+1+q)).$$

Therefore we have, when $q > 2$,

$$D_{q-1}(x) = \frac{x^{q-1}}{4\Gamma(q)} + \frac{x^q}{\Gamma(1+q)}(\gamma + \log x - \psi(q+1)) + 2\pi x^q \sum_{n=1}^{\infty} d(n)\lambda_q(4\pi\sqrt{nx}), \tag{3.26}$$

where

$$\lambda_q(z) = -\frac{2}{\pi}\sum_{m=0}^{\infty} \frac{(z/2)^{4m}}{\Gamma(2m+1)\Gamma(2m+1+q)}(2\log(z/2) - \psi(2m+1) - \psi(2m+1+q)) \tag{3.27}$$

is the "generalized Bessel function". To see that this terminology is justified, note that from (3.6) and (3.7) it follows that

$$K_1(2z) + \frac{\pi}{2}Y_1(2z) = z\sum_{m=0}^{\infty}\frac{z^{4m}}{(2m)!\,(2m+1)!}(2\log z - \psi(2m+1) - \psi(2m+2)),$$

so that a comparison with (3.27) shows that

$$\lambda_1(4\pi\sqrt{nx}) = -\pi^{-2}(xn)^{-1/2}(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})). \tag{3.28}$$

Therefore the main effort must be directed towards showing that (3.26) holds not only for $q > 2$, but for $q \geq 1$, since for $q = 1$ (3.26) reduces to (3.1) in view of (3.28), proving Voronoï's formula (3.1) for $\Delta(x)$ when x is not an integer.

The definition of λ_q as an integral shows that

$$\frac{2}{\pi}\lambda_q(z) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \frac{(z/2)^{2s-2}\,ds}{\Gamma(s)\Gamma(s+q)\cos^2 \pi s/2} \tag{3.29}$$

for $\operatorname{Re} q > 0$, $-1 < c < 1$, $\operatorname{Re} q + 2c > 2$. Using a technique similar to the one used in the proof of (3.16) it is seen that the integral in (3.29) may be asymptotically evaluated to yield, for $-\pi/2 < \arg z < 3\pi/2$,

$$\lambda_q(z) \sim (z/2)^{-q}(-Y_q(z) + \frac{2}{\pi}e^{q\pi i}K_q(z)) - \frac{2}{\pi}\sum_{r=1}^{\infty} \frac{\Gamma(2r)}{\Gamma(q-2r+1)}(2/z)^{4r}. \tag{3.30}$$

Hence by the asymptotic formulas (3.12) and (3.13)

$$\lambda_q(4\pi\sqrt{nx}) = (nx)^{-q/2-1/4}\left\{A_q\sin(4\pi\sqrt{nx} - \pi/4 - \pi q/2) + B_q e^{-4\pi\sqrt{nx}}\right\} + O((nx)^{-q/2-5/4}) + O((nx)^{-2}), \tag{3.31}$$

where A_q and B_q are uniformly bounded. Therefore the series in (3.26) converges absolutely for $\operatorname{Re} q > 3/2$, and this establishes the validity of (3.26) in the range $q > 3/2$. To investigate (3.26) for $q \leq 3/2$ one needs to know the behaviour of the partial sums of the series in (3.26). Letting

$$r_0(x) = D_0(x) - x(\gamma + \log x - \Psi(2)) - 1/4,$$

$$r_1(x) = \int_0^x r_0(t)\,dt = \sum_{n \leq x} (x-n)d(n) - \tfrac{1}{2}x^2(\gamma + \log x - \Psi(3)) - x/4,$$

we obtain for $a, b > 0$, $f(x) \in C^2[a,b]$

$$\sum_{a<n\leq b} f(n)d(n) = \int_a^b f(t)\,dD_0(t) = \int_a^b f(t)\,dr_0(t) + \int_a^b (2\gamma + \log t)f(t)\,dt,$$

and integrating twice by parts we obtain

$$(3.32) \qquad \sum_{a<n\leq b} f(n)d(n) = (r_0(t)f(t) - r_1(t)f'(t))\Big|_a^b + \int_a^b r_1(t)f''(t)\,dt + \int_a^b (2\gamma + \log t)f(t)\,dt$$

From (3.26) and (3.31) it is seen that $r_1(x) \ll x^{3/4}$, while trivially one has $r_0(x) \ll x^{1/2}$. To establish the convergence of (3.26) for $1 \leq q \leq 3/2$ we use (3.32) with $f(t) = \lambda_q(4\pi\sqrt{xt})$ and note that from the series expansion (3.27) we obtain for integral q

$$(3.33) \qquad \frac{d}{dt}(t^q \lambda_q(4\pi\sqrt{nt})) = t^{q-1}\lambda_{q-1}(4\pi\sqrt{nt}),$$

which is analogous to (3.11). By some calculations it follows that

$$f''(t) \sim Ax^{3/4-q/2}t^{-q/2-5/4}\sin(4\pi\sqrt{xt} - q\pi/2 - 5\pi/4) + O(x^{-2}t^{-4}),$$

and using again $r_0(x) \ll x^{1/2}$, $r_1(x) \ll x^{3/4}$ it is seen from (3.32) that the series in (3.26) converges for $q \geq 1$, and moreover when x is not an integer the convergence is uniform for x lying in any closed interval free of integers for $q > 1/2$. A more careful analysis, based on investigation of the function $f(t) = \lambda_1(4\pi\sqrt{xt}) - \lambda_1(4\pi\sqrt{mt})$, $m = [x]$, settles the case $q = 1$, x is an integer. The details may be found in the work of Dixon and Ferrar [1].

Finally it remains to discuss the proof of the summation formula (3.2). Note first that using (3.1) and (3.11) one obtains formally

$$\sum_{a<n\leq b} f(n)d(n) = \int_a^b f(x)\,dD(x) = \int_a^b (\log x + 2\gamma)f(x)\,dx + \int_a^b f(x)\,d\Delta(x) =$$

$$\int_a^b (\log x + 2\gamma) f(x)\,dx \quad + \quad \sum_{n=1}^{\infty} d(n) \int_a^b f(x)\alpha(nx)\,dx,$$

i.e. one obtains formally (3.2) from (3.1) by differentiating $\Delta(x)$ term by term, but this procedure is hard to justify. A rigorous proof may be based on (3.26) and the summation formula (3.32), when we substitute

$$r_1(t) = 2\pi t^2 \sum_{n=1}^{\infty} d(n)\lambda_2(4\pi\sqrt{nt}).$$

Since $f''(t)$ is bounded and the series for $r_1(t)$ is absolutely and uniformly convergent the order of summation and integration may be inverted, and the first integral in (3.32) becomes

$$2\pi \sum_{n=1}^{\infty} d(n) \int_a^b t^2\lambda_2(4\pi\sqrt{nt}) f''(t)\,dt.$$

Integrating twice by parts and using (3.33) we have

$$\int_a^b t^2\lambda_2(4\pi\sqrt{nt}) f''(t)\,dt = \left(t^2\lambda_2(4\pi\sqrt{nt}) f'(t) - t\lambda_1(4\pi\sqrt{nt}) f(t)\right)\Big|_a^b + \int_a^b \lambda_0(4\pi\sqrt{nt}) f(t)\,dt. \tag{3.34}$$

From (3.4), (3.5) and (3.27) it is seen that

$$2\pi\lambda_0(4\pi\sqrt{nx}) = 4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx}),$$

and using (3.26) with $q = 1$ and $q = 2$ we have (in view of (3.1) and (3.24))

$$r_0(t) = \tfrac{1}{2}d(t) + 2\pi t \sum_{n=1}^{\infty} d(n)\lambda_1(4\pi\sqrt{nt}),\; r_1(t) = 2\pi t^2 \sum_{n=1}^{\infty} d(n)\lambda_2(4\pi\sqrt{nt}),$$

where we set $d(t) = 0$ if t is not an integer. Therefore if we multiply (3.34) by $2\pi d(n)$ and sum over n we obtain (3.2) from (3.32).

§4. Other formulas of the Voronoï type

There exists a large literature concerning various generalizations of Voronoï's formulas (3.1) and (3.2) to other arithmetical functions, whose generating functions satisfy functional equations similar to the functional equation for the zeta-function of Riemann. This possibility of generalizations is one of the most important aspects of the research initiated by Voronoĭ, but since our main purpose is the investigation concerning the zeta-function, we shall mention only one result explicitly which is similar to Voronoĭ's formulas. This concerns the classical lattice-point problem known as the circle problem, which is similar to the divisor problem (see Chapter 10) and consists of estimating the function

$$(3.35) \qquad P(x) = R(x) - \pi x - 1 = \sum_{n\le x}{}' r(n) - \pi x - 1,$$

where $r(n)$ is the number of ways n may be written as a sum of two integer squares. In 1916 G.H. Hardy [3] proved the asymptotic expansion

$$(3.36) \quad \frac{1}{\Gamma(q)}\sum_{n\le x}{}'(x-n)^{q-1}r(n) = \frac{\pi x^q}{\Gamma(q+1)} - \frac{x^{q-1}}{\Gamma(q)} + \pi^{1-q}x^{q/2}\sum_{n=1}^{\infty} n^{-q/2}r(n)J_q(2\pi\sqrt{nx})$$

for $q \ge 1$ (here $\sum'$ means that only for $q = 1$ and $n = x$ one should take $r(n)/2$ instead of $r(n)$). From Hardy's formula one may derive a summation formula for $\sum_{a\le n\le b}{}' f(n)r(n)$ analogous to (3.2). The expression on the right-hand side of (3.36) is simpler than the corresponding one in (3.26), and (3.36) may be deduced more simply than (3.26), since the generating series of $r(n)$ has a simple pole at $s = 1$, while $\zeta^2(s)$ (which is the generating function of $d(n)$) has a second order pole at $s = 1$.

NOTES

G.F. Voronoĭ proved by a complicated method the formulas (3.1) and (3.2) in [1], and a little later in [2] he succeeded in generalizing his method to certain other functions which are the number of representations of n by certain positive-definite quadratic forms. The methods introduced by Voronoĭ were deep and inspired much subsequent research, of which one example was mentioned in §4. Modern developments of this theory may be found in the works of K. Chandrasekharan and R. Narasimhan [1], [2], B.C. Berndt [1], [2], [3], [4] and J.L. Hafner [2].

The notation used in this chapter differs a little from the notation used in other chapters in two instances, but this will cause hopefully no confusion. Firstly the error term $\Delta(x) = \Delta_2(x)$ is defined somewhat differently than in Chapter 10, and secondly following traditional notation we have defined in (3.8)

$\psi(z) = \Gamma'(z)/\Gamma(z)$, while in the rest of this text we use $\psi(x) = x - [x] - 1/2$; the function $\psi(x)$ (see Notes in Chapter 9) is also used as $\psi(x) = \sum_{n\le x}\Lambda(n)$ in prime number theory.

When p is an integer in (3.9) $\Gamma(p+k+1)$ is undefined when $p + k + 1$ is a non-positive integer, but for integer values of p it is clear that one should define

$$J_p(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{p+2k}}{k!\,(p+k)!} \quad (p \ge 0),\ J_{-p}(z) = (-1)^p J_p(z),$$

and similarly for $I_p(z)$ when p is an integer.

There is a possibility of obtaining an explicit expression for $\Delta(x)$ which is completely different from (3.1). Namely starting from the elementary expression

$$\sum_{n\le x} d(n) = 2\sum_{n\le\sqrt{x}}\ \sum_{m\le x/n} 1 - [\sqrt{x}]^2$$

and defining $\Delta(x)$ as in (3.1), a simple calculation gives at once

$$\Delta(x) = -2\sum_{n\le\sqrt{x}}\psi(x/n) + O(x^{\varepsilon}),\ \psi(x) = x - [x] - 1/2.$$

This is a useful formula, but for most purposes (3.1) and the flexible (3.17) are better.

All the facts used here about the Bessel functions may be found in the standard work of G.N. Watson [1]. Curiously enough, Watson mentions Voronoï's formula only once on p. 200, where he writes rather disparagingly : "A novel application of these asymptotic expansions has been discovered in recent years: they are of some importance in the analytic theory of the divisors of numbers". In view ofmany important applications of (3.1) and (3.2) and allthe research Voronoï's work has inspired, this remark seems a little unjust - Voronoï's formulas deserve more than a casual mention.

Titchmarsh's proof of (3.16) is given in Chapter 12 of [8] and some details of the proof are for this reason suppressed here. However his remark on p. 268, which amounts to saying that (3.20) holds, is rather casual. The reference to (7.9.8) and (7.9.11) in his book [7] on Fourier integrals does not seem adequate, and it is desirable to have a more detailed account of (3.20). To see that (3.21) and (3.22) hold, note that $f(x) = x^{a-1}$ $(0 < a < 1)$ and $F_c(x) = \sqrt{2/\pi}\,\Gamma(a)x^{-a}\cos\pi a/2$ are cosine transforms, i.e.

$$F_c(x) = \sqrt{2/\pi}\int_0^\infty f(t)\cos xt\cdot dt.$$

By the definition of $J_p(x)$

$$\int_0^1 (1-y^2)^{p-1/2}\cos xy\cdot dy = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}\int_0^1 (1-y^2)^{p-1/2}y^{2n}dy =$$

$$\frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n x^{2n}\Gamma(p+1/2)\Gamma(n+1/2)}{(2n)!\,\Gamma(p+n+1)} = 2^{p-1/2}\sqrt{\pi/2}\,\Gamma(p+1/2)x^{-p}J_p(x),$$

where we used (1.29) and $\Gamma(n+1/2) = 2^{-n}\pi^{1/2}(2n-1)!!$. Thus

$$f(x) = \begin{cases} (1-x^2)^{p-1/2}, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}, \quad F_c(x) = 2^{p-1/2}\Gamma(p+1/2)x^{-p}J_p(x)$$

are also cosine transforms. Analogously to (1.6) one obtains

$$\int_0^\infty F_c(x)G_c(x)dx = \int_0^\infty f(x)g(x)dx, \quad \int_0^\infty F_c(x)g(x)dx = \int_0^\infty G_c(x)f(x)dx$$

for two pairs $f(x), F_c(x)$ and $g(x), G_c(x)$ of cosine trabsforms, which gives

$$(3.37) \qquad \int_0^\infty J_p(x)x^{a-p-1}dx = \sqrt{2/\pi}\frac{\Gamma(a)\cos\pi a/2}{2^{p-1/2}\Gamma(p+1/2)}\int_0^1 (1-x^2)^{p-1/2}x^{-a}dx =$$

$$\frac{\Gamma(a)\cos\pi a/2}{2^{p-1}\sqrt{\pi}\,\Gamma(p+1/2)}\cdot\frac{\Gamma(p+1/2)\Gamma(1/2-a/2)}{2\Gamma(p-a/2+1)} = \frac{2^{a-p-1}\Gamma(a/2)}{\Gamma(p-a/2+1)},$$

where we used (1.28). Taking $a = s$ to be complex in (3.37) it is seen that (3.37) holds for $0 < \delta < p + 3/2$ in view of (3.14), and we have the Mellin transforms

$$(3.38) \qquad x^{-p}J_p(x), \quad \frac{2^{s-p-1}\Gamma(s/2)}{\Gamma(p-s/2+1)}.$$

Finally using the relations

$$Y_p(x) = \frac{J_p(x)\cos\pi p - J_{-p}(x)}{\sin\pi p}, \quad K_p(x) = \frac{\pi i}{2}e^{p\pi i/2}(J_p(x) + iY_p(x))$$

one obtains (3.21) and (3.22) easily from (3.38).

Both the proof of (3.1) and (3.16) utilize the familiar functional equation for the zeta-function, which belongs to elementary zeta-function theory and is supposed to be known to the reader. However for the sake of completeness a proof of the functional equation will be presented in §1 of Chapter 4.

The method of considering the weighted divisor sum $D_{q-1}(x)$ is due to Dixon and Ferrar [1], while their paper [2] contains an interesting investigation of a reciprocity relation connected with the Voronoï formula, which is motivated by the reciprocity relation for Fourier transforms. The exposition presented in §3 concerning the proof of (3.1) and (3.2) follows Dixon and Ferrar [1], where additional details (like the proof of (3.30), and especially the proof of (3.1) when x is an integer) may be found. The main idea in the proof of Dixon and Ferrar is to prove (3.26) for some q (specifically for $q > 3/2$), and then to feed back (3.26) to itself again (in a certain sense) via the summation formula (3.32) to obtain (3.26) for values of q less than 3/2 also. The crucial point in their proof of (3.2) is the fact that the expression for $r_1(t)$ allows one to invert the order of summation and integration - the rest is simply integration by parts. The paper of Dixon and Ferrar [1] gives also an analysis of (3.2) when $a = 0$ and $b = \infty$, in which case there are some additional difficulties. A proof of (3.2) when $a = 0$, $b = \infty$ has been given recently by D. Hejhal [1] who used a two-dimensional Poisson summation formula.

A nice generalization of the truncated formula (3.17) for $\Delta(x)$ to the error term in the asymptotic formula for $\sum_{n\le x}{}' f(n)(x-n)^{\varkappa}, \varkappa \ge 0$ has been made by H.-E. Richert [2]. Here f(n) is an arithmetical function generated by a Dirichlet series which satisfies a certain type of functional equation involving gamma factors, which is similar to the ordinary functional equation for the Riemann zeta-function. Applications of Richert's results to the circle problem will be discussed in Chapter 10.

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TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 4

THE APPROXIMATE FUNCTIONAL EQUATIONS

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§1. The classical functional equation

The classical functional equation for the Riemann zeta-function is

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s), \tag{4.1}$$

and was originally discovered and proved by B. Riemann in his epoch-making memoir [1]. Using (1.28) one may write (4.1) as

$$\zeta(s)\Gamma(s) = (2\pi)^s\zeta(1-s)/(2\cos\pi s/2), \tag{4.2}$$

or simply as

$$\zeta(s) = \chi(s)\zeta(1-s),\ \chi(s) = (2\pi)^s/(2\Gamma(s)\cos\pi s/2). \tag{4.3}$$

The functional equation holds for all complex s, and represents one of the fundamental tools of zeta-function theory. Following traditional notation which originated with Riemann we shall write $s = \sigma + it$, σ and t real, and so using (1.32) it follows immediately that

$$\chi(s) = (2\pi/t)^{\sigma+it-1/2}e^{i(t+\pi/4)}(1 + O(t^{-1})),\ t \ge t_0, \tag{4.4}$$

which is for most purposes a sufficiently sharp approximation. Though there exist many well-known ways in which (4.1) or one of its equivalents may be proved, a proof of the functional equation will be given now for the sake of completeness of the exposition. The proof has the advantage of being almost elementary, and starts from the identity

$$(1 - e^{-x2^{-n}})\prod_{k=1}^{n}(1 + e^{-x2^{-k}}) = 1 - e^{-x}.$$

Therefore by logarithmic differentiation we obtain, for $x > 0$,

$$\sum_{k=1}^{n}\frac{-2^{-k}e^{-x2^{-k}}}{1 + e^{-x2^{-k}}} = \frac{e^{-x}}{1 - e^{-x}} - \frac{2^{-n}e^{-x2^{-n}}}{1 - e^{-x2^{-n}}},$$

or

$$\sum_{k=1}^{n}\frac{2^{-k}}{e^{x2^{-k}} + 1} = \frac{1}{x}\cdot\frac{x2^{-n}}{e^{x2^{-n}} - 1} - \frac{1}{e^x - 1}. \qquad (x > 0)$$

Letting $n \to \infty$ we obtain the identity

$$(4.5) \qquad \sum_{k=1}^{\infty} \frac{2^{-k}}{e^{x2^{-k}}+1} = \frac{1}{x} - \frac{1}{e^x - 1}, \qquad (x > 0)$$

which is the starting point for our proof of the functional equation. Consider now for $0 < \sigma < 1$

$$\int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1}\,dx = \sum_{k=1}^{\infty} 2^{-k} \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} e^{-n2^{-k}x} x^{s-1}\,dx =$$

$$\Gamma(s) \sum_{k=1}^{\infty} 2^{-k} 2^{ks} \sum_{n=1}^{\infty} (-1)^n n^{-s} = \Gamma(s) \frac{2^{s-1}}{1 - 2^{s-1}} (2^{1-s} - 1)\zeta(s) = \Gamma(s)\zeta(s).$$

Here (4.5) was used, the elementary representation

$$(2^{1-s} - 1)\zeta(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}, \qquad (0 < \sigma < 1)$$

and the fact that the order of summation and integration may be inverted by absolute convergence. Change of variable $\sqrt{2\pi}y = x$ gives now for $0 < \sigma < 1$

$$(4.6) \qquad F(s) = \zeta(s)\Gamma(s)(2\pi)^{-s/2} = \int_0^{\infty} f(x)x^{s-1}\,dx,$$

where

$$(4.7) \qquad f(x) = (e^{\sqrt{2\pi}x} - 1)^{-1} - (\sqrt{2\pi}x)^{-1}.$$

If we now use the fact that $f(x)$ is self-reciprocal with respect to sine transforms, i.e.

$$(4.8) \qquad f(t) = (2/\pi)^{1/2} \int_0^{\infty} f(x) \sin xt \cdot dx,$$

then the proof of (4.2) easily follows from (4.6), which shows that $F(s)$ is the Mellin transform of $f(x)$. Namely with (4.6) we have

$$(4.9) \qquad \begin{aligned} F(s) &= \int_0^{\infty} f(x)x^{s-1}\,dx = (2/\pi)^{1/2} \int_0^{\infty} f(y) \left(\int_0^{\infty} x^{s-1} \sin xy \cdot dy \right) dx = \\ &(2/\pi)^{1/2}\Gamma(s) \sin \pi s/2 \cdot \int_0^{\infty} f(y) y^{-s}\,dy = (2/\pi)^{1/2}\Gamma(s)F(1-s)\sin \pi s/2 = F(s), \end{aligned}$$

and the last equality also holds by analytic continuation outside the strip $0 < \sigma < 1$. Using the first identity in (1.28) we obtain finally from (4.9)

$$\zeta(s)\Gamma(s)(2\pi)^{-s/2} = (2/\pi)^{1/2}\sin(\pi s/2)\Gamma(s)\zeta(1-s)(2\pi)^{(s-1)/2} = \frac{\zeta(1-s)(2\pi)^{s/2}}{2\cos(\pi s/2)},$$

which is exactly (4.2).

§2. Approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$

There exists a large number of the so-called "approximate functional equations" for $\zeta^k(s)$, which express $\zeta^k(s)$ by one or more finite sums whose lengths depend on $|t|$. In this section we shall examine the simplest cases when $k = 1$ and $k = 2$. The classical results on this subject are due to G. H. Hardy and J.E. Littlewood [2],[3], A.E. Ingham [1] and E.C. Titchmarsh [5]. These are

$$\zeta(s) = \sum_{n\le x} n^{-s} + \chi(s)\sum_{n\le y} n^{s-1} + O(x^{-\delta}) + O(t^{1/2-\delta}y^{\delta-1}), \tag{4.10}$$

which is valid for $0 < \delta < 1$; $2\pi xy = t$; $x,y,t > C > 0$ and

$$\zeta^2(s) = \sum_{n\le x} d(n)n^{-s} + \chi^2(s)\sum_{n\le y} d(n)n^{s-1} + O(x^{1/2-\delta}\log t), \tag{4.11}$$

which is valid for $0 < \delta < 1$; $4\pi^2xy = t^2$; $x,y,t > C > 0$. Because of symmetry a corresponding result holds also if $t < 0$ with t replaced by $|t|$ in the error terms and in $2\pi xy = t$ for (4.10). The approximate functional equations (4.10) and (4.11) possess a symmetric property if $s = 1/2 + it$. Namely from (4.3) it is seen that $\chi(1/2 + it) = \chi^{-1}(1/2 - it)$, so that $\zeta(1/2 + it)\chi^{-1/2}(1/2 + it)$ is real, and therefore (4.10) and (4.11) with $x = y = (t/2\pi)^{1/2}$ and $x = y = t/2\pi$ respectively yield

$$\zeta(1/2+it)\chi^{-1/2}(1/2+it) = 2\mathrm{Re}\Big\{\chi^{1/2}(1/2+it)\sum_{n\le(t/2\pi)^{1/2}} n^{-1/2+it}\Big\} + O(t^{-1/4}), \tag{4.12}$$

$$|\zeta(1/2 + it)|^2 = \zeta^2(1/2 + it)\chi^{-1}(1/2 + it) = \tag{4.13}$$

$$2\mathrm{Re}\Big\{\chi(1/2 + it)\sum_{n\le t/2\pi} d(n)n^{-1/2+it}\Big\} + O(\log t),$$

since $|\chi(1/2 + it)| = 1$.

The proof of the well-known relation (4.10) may be carried out via the Poisson summation formula (1.23), but rather than to do this here we shall present now a proof of the more difficult (4.11) by using the Voronoï summation formula (in fact it will be technically simpler in (4.15) to use (3.1) instead of (3.2)), which may be in a certain sense considered as a two-dimensional Poisson summation formula. The proof of (4.11) will be given now for $\sigma \geq 1/2$, $x \geq y > t^{\varepsilon}$, and rather than to try to adapt the proof to cover the range $0 < \sigma < 1/2$, in §3 the classical method of Hardy and Littlewood [3] will be presented in conjuction with the approximate functional equation for $\zeta^k(s)$, which enables one to obtain a complete and different proof of (4.11).

With $D(x) = \sum_{n\leq x} d(n) = x(\log x + 2\gamma - 1) + \Delta(x), \sigma > 1$, we have

$$\zeta^2(s) = \sum_{n\leq N} d(n)n^{-s} + \int_{N+0}^{\infty} x^{-s}dD(x) = \sum_{n\leq N} d(n)n^{-s} + \int_N^{\infty}(\log x+2\gamma)x^{-s}dx + \int_N^{\infty} x^{-s}d\Delta(x).$$

As discussed in §2 of Chapter 3, a trivial consequence of the Voronoï formula (3.17) is the order estimate $\Delta(x) \ll x^{1/3+\varepsilon}$. Thus an integration by parts gives

(4.14) $$\zeta^2(s) = \sum_{n\leq N} d(n)n^{-s} + (s-1)^{-1}N^{1-s}(\log N + 2\gamma) + (s-1)^{-2}N^{1-s} + O(N^{\varepsilon+1/3-\sigma}) +$$

$$+ s\int_N^{\infty} x^{-s-1}\Delta(x)dx.$$

The integral in (4.14) is therefore seen to be absolutely convergent for $\sigma > 1/3$ (and using Theorem 10.5 it is seen that the integral in question is actually absolutely convergent in a wider semi-plane $\sigma > 1/4$), so that (4.14) furnishes an analytic continuation of $\zeta^2(s)$ for $\sigma > 1/3$. Our choice for N will be $N = t^c$, where $c > 0$ is fixed but sufficiently large. We use (3.1) and split the series involving $d(n)$ at $(1+\varepsilon)y$, where $xy = (t/2\pi)^2$. Integration by parts gives

(4.15) $$\sum_{x<n\leq N} d(n)n^{-s} = \int_x^N(\log u + 2\gamma)u^{-s}du + O(x^{1/2-\sigma}\log t) +$$

$$\sum_{n\leq(1+\varepsilon)y} d(n)\int_x^N (4K_0(4\pi\sqrt{nu})-2\pi Y_0(4\pi\sqrt{nu}))u^{-s}du +$$

$$s\sum_{n>(1+\varepsilon)y} d(n)\int_x^N (-2\pi^{-1}(u/n)^{1/2}(K_1(4\pi\sqrt{nu})+ \tfrac{\pi}{2}Y_1(4\pi\sqrt{nu})))u^{-s-1}du.$$

Noting that

$$\int_x^N(\log u + 2\gamma)u^{-s}du = (s-1)^{-1}x^{1-s}(\log x + 2\gamma) + (s-1)^{-2}x^{1-s} -$$

$$-(s-1)^{-1}N^{1-s}(\log N + 2\gamma) - (s-1)^{-2}N^{1-s},$$

it follows on comparing (4.14) and (4.15) that the only difficulty lies in the estimation of the sums appearing on the right-hand sides of (4.15). Using (3.12) and (3.13) it is seen that the second sum on the right-hand side of (4.15) is equal to $c_1 s$ times

$$(4.16)\qquad \sum_{n>(1+\varepsilon)y} d(n)\Big\{n^{-3/4}\int_x^N u^{-s-3/4}\sin(4\pi\sqrt{nu}-3\pi/4)\,du + O\Big(\int_x^N n^{-7/4}u^{-\delta-7/4}du\Big)\Big\} =$$

$$\pm(2i)^{-1}\sum_{n>(1+\varepsilon)y} d(n)n^{-3/4}\int_x^N u^{-\delta-3/4}\exp(-it\log u \pm 4\pi\sqrt{nu}\mp 3\pi/4)\,du + O(x^{-\delta-3/4}y^{-3/4}\log t),$$

and the error term above is $\ll t^{-1}x^{1/2-\delta}\log t$. The integrals are of the form

$$\int_x^N u^{-\delta-3/4}e(F(u))\,du,\quad F(u) = -\frac{t}{2\pi}\log u \pm 2\sqrt{nu},$$

so that F is monotonic and $|F'(u)| \gg (n/x)^{1/2}$ for $x \le u \le N$ in view of $n > (1+\varepsilon)y$. Using (2.3) it is then seen that the total contribution of the sum with $n>(1+\varepsilon)y$ is

$$\ll x^{1/2-\delta}\log t + t\sum_{n>(1+\varepsilon)y} d(n)n^{-3/4}n^{-1/2}x^{-\delta-1/4} \ll$$

$$x^{1/2-\delta}\log t + ty^{-1/4}x^{-\delta-1/4}\log t \ll x^{1/2-\delta}\log t,$$

since $t \ll x$ if $x \ge y$ and $xy = (t/2\pi)^2$. Setting for brevity $T = t/2\pi$ and using again (3.12) and (3.13) it is seen that the first sum on the right-hand side of (4.15) is equal to

$$-2^{1/2}\sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}\int_x^N u^{-\delta-1/4}\exp(-it\log u)\sin(4\pi\sqrt{nu}-\pi/4)\,du +$$

$$+ O\Big(\sum_{n\le(1+\varepsilon)y} d(n)n^{-3/4}\int_x^N u^{-\delta-3/4}du\Big) =$$

$$\mp 2^{-1/2}i^{-1}\sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}\int_x^N u^{-\delta-1/4}e(-T\log u \pm 2\sqrt{nu}\mp 1/8)\,du + O(y^{1/4}x^{1/4-\delta}\log t),$$

and clearly $y^{1/4}x^{1/4-\delta} \ll x^{1/2-\delta}$. The integral with $e(-T\log u - 2\sqrt{nu}+1/8)$ is as in the previous case estimated by (2.3), the total contribution of these integrals being now

$$\ll \sum_{n\le(1+\varepsilon)y} d(n)n^{-3/4}x^{1/4-\delta} \ll y^{1/4}x^{1/4-\delta}\log t \ll x^{1/2-\delta}\log t.$$

Therefore we are left with

$$(4.17)\qquad -2^{-1/2}i^{-1}\sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}I_n,\ I_n = \int_x^N u^{-\delta-1/4}e(f_n(u))du,$$

$$f_n(u) = -T\log u + 2\sqrt{nu} - 1/8.$$

Thus we have

$$f_n'(u) = -Tu^{-1} + (n/u)^{1/2},\ f_n''(u) = Tu^{-2} - \tfrac{1}{2}n^{1/2}u^{-3/2},$$

implying $f_n''(u) > 0$ for $u \le u_o = 4T^2n^{-1}$. But for $u \ge (1-\varepsilon)u_o$ we see that $f_n'(u) \gg (n/u)^{1/2}$, and so using (2.3)

$$\sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}\int_{(1-\varepsilon)u_o}^{N} u^{-\delta-1/4}e(f_n(u))du \ll \sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}u_o^{1/4-\delta}n^{-1/2}$$

$$\ll \sum_{n\le(1+\varepsilon)y} t^{1/2-2\delta}d(n)n^{\delta-1} \ll t^{1/2-2\delta}y^{\delta}\log t \ll x^{1/2-\delta}\log t,$$

if $\varepsilon > 0$ is a sufficiently small fixed number. For the integrals in the remaining sum

$$-2^{-1/2}i^{-1}\sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}\int_x^{(1-\varepsilon)u_o} u^{-\delta-1/4}e(f_n(u))du$$

we shall use Theorem 2.2 with $a = x$, $b = (1-\varepsilon)u_o$, $\phi(u) = u^{-\delta-1/4}$, $f(u) = f_n(u)$, $k = 0$, $F = \mu = t$, and the conditions of the theorem are readily checked. All the error terms in Theorem 2.2 are easily seen to contribute a total $\ll x^{1/2-\delta}\log t$, except the error term $O(\Phi_a(|f_a' + k| + f_a''^{\,1/2})^{-1})$, which will be discussed now. Observe that for a given n we have $f_n'(x) = 0$ if $n = T^2x^{-1} = y$ and y is an integer. Therefore making the substitution $n = [y] + m$, $|m| \le \varepsilon y$ we have

$$f_n'(x) = f_n'(x) - f_y'(x) \asymp |m|t^{-1}$$

by the mean value theorem, so that

$$(4.18)\qquad (|f_n'(x)| + (f_n''(x))^{1/2})^{-1} \ll \begin{cases} xt^{-1/2} & \text{for } |m| \le T^2x^{-2} \\ |m|^{-1}t \quad \text{for } t^2x^{-2} \ll |m| \le \varepsilon y/2 \\ \max((n/x)^{-1/2}, x/t) & \text{for } |m| > \varepsilon y/2. \end{cases}$$

In view of $T^2x^{-2} \ll 1$ the first estimate in (4.18) can hold for at most $O(1)$ values of m, and the total contribution of the error term

$$O(\Phi_a(|f_a' + k| + f_a''^{\,1/2})^{-1})$$

is then

$$\sum_{|m| \ll t^2x^{-2}} t^{\varepsilon} y^{-1/4} x t^{-1/2} x^{-\delta-1/4} + \sum_{1 \ll |m| \le \varepsilon y/2} t|m|^{-1} d([y]+m)([y]+m)^{-1/4} x^{-\delta-1/4}$$

$$(4.19)\qquad + \sum_{\varepsilon y/2 \le |m| \le y} x^{1/2} d([y]+m)([y]+m)^{-3/4} x^{-\delta-1/4} +$$

$$+ \sum_{\varepsilon y/2 \le |m| \le y} xt^{-1} d([y]+m)([y]+m)^{-1/4} x^{-\delta-1/4} \ll t^{1/2+\varepsilon} x^{-\delta} + x^{1/2-\delta}\log t,$$

and where in the second sum in (4.19) we used the trivial $d(n) \ll n^{\varepsilon}$. Here the error term $t^{1/2+\varepsilon}x^{-\delta}$ does not exceed $x^{1/2-\delta}\log t$ if $x > t^{1+2\varepsilon}$, but it has been kindly pointed out to me by M. Jutila that by a more elaborate consideration of the error term $\Phi_a(|f_a' + k| + f_a''^{\,1/2})^{-1}$ in Atkinson's Theorem 2.2 one can obtain that the contribution of the sums in (4.19) is indeed $\ll x^{1/2-\delta}\log t$ for the whole range $x \gg t$.

Finally it remains to deal with the main terms, i.e. the saddle point terms coming from Theorem 2.2 and then to use (4.4). The only root of $f_n'(u) = 0$ for a fixed n is $x_0 = T^2n^{-1}$, and $x_0 > x$ precisely if $n \le y = T^2x^{-1}$. Now

$$f_n''(x_0) = \tfrac{1}{2}T^{-3}n^2,$$

$$f_n(x_0) = -T\log(T^2n^{-1}) + 2T - 1/8,$$

so that we have

$$-2^{-1/2}i^{-1}d(n)n^{-1/4}\varphi(x_0)f_n''(x_0)^{-1/2}\exp(2\pi i f_n(x_0) + \pi i/4) =$$

$$-2^{-1/2}i^{-1}d(n)n^{-1/4}(T^2n^{-1})^{-\delta-1/4}2^{1/2}T^{3/2}n^{-1}\exp(2it\log(2\pi/t)+it\log n+2it+\pi i/2)\exp(-\pi i/2)=$$

$$= d(n)n^{\delta-1+it}\chi^2(\delta + it) + O(d(n)n^{\delta-1}t^{-2\delta}).$$

Therefore

$$-2^{-1/2}i^{-1}\sum_{n\le(1+\varepsilon)y} d(n)n^{-1/4}I_n = \sum_{n\le y} d(n)n^{\delta-1+it}\chi^2(\delta + it) +$$

$$+ O(t^{-2\delta}\sum_{n\le 2y} d(n)n^{\delta-1}) + O(x^{1/2-\delta}\log t) =$$

$$\chi^2(s)\sum_{n\le y} d(n)n^{s-1} + O(x^{1/2-\delta}\log t),$$

and in view of (4.17) this means that we have proved the approximate functional equation (4.11).

§3. The approximate functional equation for higher powers

We pass now to the analogues of (4.10) and (4.11) for $\zeta^k(s)$, where $k \ge 3$ is a fixed integer. The approach that will be used is that of R. Wiebelitz [1], and is based on Hardy and Littlewood's proof [3] of the approximate functional equation for $\zeta^2(s)$, so that this method yields an alternative proof of (4.11) for $0 < \delta < 1$; $4\pi^2xy = t^2$; $x,y,t > C > 0$, but it seemed interesting to treat the important case $k = 2$ by Voronoï's formula also. The proof will use a certain "Tauberian" argument (essentially recovering $\sum_{n\le x} d_k(n)n^{-s}$ from the weighted sum $\sum_{n\le x} d_k(n)n^{-s}(\log x/n)^{k-1}$) and estimates for power moments of the zeta-function, which will be extensively discussed in later chapters (and which do not depend on results of this section). New estimates for power moments of the zeta-function on the critical line lead to overall improvements of Wiebelitz's results, but as k grows the order of the error terms in the approximate functional equation becomes rather large, which is to be only expected, and thus for practical reasons the detailed analysis is carried out only for $k \le 12$.

For simplicity of writing we shall use the notation

$$(4.20)\qquad X(s) = \chi^k(s),\ \log T = -\frac{X'(1/2 + it)}{X(1/2 + it)} = -k\frac{\chi'(1/2 + it)}{\chi(1/2+ it)},$$

and furthermore as in the proof of (4.11) we shall suppose that $t > 0$. By (4.1) we have

$$\frac{\chi'(s)}{\chi(s)} = \log\pi - \frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - \frac{s}{2}\right) - \frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}(s),$$

so that for $s = 1/2 + it$ the above equation shows that T, as defined by (4.20), is real and moreover using (1.33) we have

$$-\frac{\chi'(1/2 + it)}{\chi(1/2 + it)} = -\log 2\pi + \log t + O(t^{-2}),$$

and this gives

$$T = (t/2\pi)^k + O(t^{k-2}). \tag{4.21}$$

Further we suppose $xy = (t/2\pi)^{k}$, $0 < \delta < 1$, and define

$$R_x(s) = \frac{x^{1-s}}{1-s}\sum_{\nu=0}^{k-1}\sum_{\rho=\nu+1}^{k} a_{-\rho,k}\, r_{\rho,\nu}\,(1-s)^{1+\nu-\rho}\log^{\nu} x, \tag{4.22}$$

where $a_{j,k}$ is the coefficient of $(s-1)^j$ in the Laurent expansion of $\zeta^k(s)$ at $s = 1$ and

$$r_{j,m} = (m!)^{-1}\sum_{i=0}^{j-m-1}(-1)^i\binom{k+i-1}{i}\binom{k-1}{j-i-m-1}.$$

Therefore in general we have

$$R_x(s) + \chi^k(s)R_y(1-s) \ll x^{1/2-\delta}t^{-1}(x+y)^{1/2}\log^{k-1}t, \tag{4.23}$$

while for $k = 3$ we may write for some absolute D

$$R_x(s) = \frac{x^{1-s}}{1-s}\left(\tfrac{1}{2}\log^2 x + 3\gamma\log x + D\right) + O(x^{1-\delta}t^{-2}\log t). \tag{4.24}$$

We shall now give a proof of the approximate functional equation for $\zeta^k(s)$ for $3 \le k \le 12$, though it has been already remarked that the method may be used both when $k = 2$ and when $k > 12$, but in the latter case the error terms tend to be large and then the improvements of Wiebelitz's results (see Notes) are small. The improvements for $3 \le k \le 12$ are however substantial, owing to a large extent to new power moments for the zeta-function on the critical line. The functional equation that will be proved is

$$\zeta^k(s) = \sum_{n\le x} d_k(n)n^{-s} + \chi^k(s)\sum_{n\le y} d_k(n)n^{s-1} - R_x(s) - \chi^k(s)R_y(1-s) + \Delta_k(x,y), \tag{4.25}$$

$0 < \delta < 1$;

where $xy = (t/2\pi)^k$; $x,y,t \gg 1$ and $\Delta_k(x,y)$ may be considered as the error term which depends on k, x and y. We shall prove that uniformly in δ

(4.26) $$\Delta_3(x,y) \ll x^{1/2-\delta}t^{1/8+\varepsilon},\ \Delta_4(x,y) \ll x^{1/2-\delta}t^{13/48+\varepsilon},$$

(4.27) $$\Delta_k(x,y) \ll t^{\varepsilon}\{x^{1/2-\delta}\min(x^{1/2},y^{1/2})t^{-2} + (x+y)^{1/2}x^{1/2-\delta}t^{-\beta} + x^{1/2-\delta}t^{(31k-52)/216}\}$$

for $5 \le k \le 12$, where $\beta = (31k - 52)/(27k - 108)$.

In case k = 3 or k = 4 one may obtain special results from (4.25) and (4.26) analogous to (4.12) and (4.13). These are

(4.28) $$\zeta^3(1/2+it)\chi^{-3/2}(1/2+it) = 2\mathrm{Re}\{\chi^{3/2}(1/2+it)\sum_{n\le(t/2\pi)^{3/2}} d_3(n)n^{-1/2+it}\} + O(t^{1/8+\varepsilon}),$$

(4.29) $$|\zeta(1/2 + it)|^4 = 2\mathrm{Re}\{\chi^2(1/2 + it)\sum_{n\le(t/2\pi)^2} d_4(n)n^{-1/2+it}\} + O(t^{13/48+\varepsilon}),$$

which follows with $s = 1/2 + it$, $x = y = (t/2\pi)^{k/2}$, as the terms with R_x and R_y are by (4.23) absorbed into the O-terms of (4.28) and (4.29) respectively.

Now we begin the proof of (4.25) by remarking that for technical reasons (as was done also by Hardy-Littlewood and Wiebelitz) the condition $xy = (t/2\pi)^k$ is replaced by $xy = T$ (see (4.20)). The error that is made in this processs is then $\ll x^{1/2-\delta}\min(x^{1/2},y^{1/2})t^{\varepsilon-2}$, which is negligible in (4.26) and present in (4.27). We shall begin the proof of the general (4.25), but at a suitable point we shall distinguish the cases k = 3, k = 4 and k > 4. For $-1/2 \le \delta \le 3/2$ we follow Wiebelitz [1] and introduce

(4.30) $$\phi_1(u) = \frac{\phi(u)}{(u - s)^2} = \frac{T^{u-s}X(u) - X(s)}{(u - s)^2},$$

so that for $\mathrm{Re}\, u \le 1/2$ and also for $\mathrm{Re}\, u < \min(\delta,1)$ the function $\phi_1(u)$ is seen to be regular and moreover uniformly in s for Re u = 1/2 we have

(4.31) $$\phi(u) \ll t^{k/2-k\delta}\min(1,t^{-1}|s-u|^2).$$

In the course of the proof we shall need the power moment estimates

(4.32) $$\int_{T-G}^{T+G} |\zeta(1/2+it)|^k dt \ll GT^{(k-4)c+\varepsilon},\ G \ge T^{2/3},\ k \ge 4,$$

where $\zeta(1/2+it) < t^{c+\varepsilon}$ (so that in view of Corollary 6.1 one may take $c = \frac{35}{216}$)

and

(4.33) $$\int_{0}^{T} |\zeta(1/2+it)|^k dt \ll T^{1+(k-4)/8+\varepsilon},\ 4 \le k \le 12.$$

The estimate (4.32) is a trivial consequence of a result of H. Iwaniec [2](see §6 of Chapter 6), while (4.33) is contained in Theorem 7.2.

The first step in the proof is to use the inversion formula (1.11) to obtain

$$I = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty} \zeta^k(s+w)x^w w^{-k}dw = \frac{1}{(k-1)!}\sum_{n\le x} d_k(n)n^{-s}\log^{k-1}x/n = S_x.$$

The line of integration is moved to $\mathrm{Re}\, w = -\gamma$, where $0 < \gamma < 3/4$, $\sigma - 1 < \gamma$, $\gamma \ne \sigma$, $\gamma \ne \sigma - 1/2$. There are poles of the integrand at $w = 0$ and $w = 1 - s$ with respective residues

$$F_x = \sum_{m=0}^{k-1} \frac{(\zeta^k(s))^{(m)}}{m!\,(k-1-m)!}(\log x)^{k-1-m}$$

and

$$Q_x = \frac{x^{1-s}}{(k-1)(1-s)^k}\sum_{m=0}^{k-1}\frac{(-1)^m(k+m-1)!}{m!\,(1-s)^m}\sum_{r=m+1}^{k} a_{-r,k}\frac{\log^{r-m-1}x}{(r-m-1)!}.$$

Hence by the residue theorem

(4.34) $$J_0 = (2\pi i)^{-1}\int_{-\gamma-i\infty}^{-\gamma+i\infty} \zeta^k(s+w)x^w w^{-k}dw = I - F_x - Q_x = S_x - F_x - Q_x.$$

Setting $s + w = z$, substituting x by T/y and using the functional equation (4.3) we obtain

(4.35) $$J_0 = X(s)(2\pi i)^{-1}\int_{\sigma-\gamma-i\infty}^{\sigma-\gamma+i\infty} \zeta^k(1-z)y^{s-z}(z-s)^{-k}dz +$$

$$+ (2\pi i)^{-1}\int_{\sigma-\gamma-i\infty}^{\sigma-\gamma+i\infty} \zeta^k(1-z)\phi(z)y^{s-z}(z-s)^{-k}dz = X(s)J_1 + J_2,$$

say. For $\sigma < \gamma$ we have by (1.11)

$$J_1 = \frac{(-1)^k}{(k-1)!} \sum_{n \le y} d_k(n) n^{s-1} (\log x/n)^{k-1} = S_y, \tag{4.36}$$

similarly to the notation already introduced in evaluating the integral I. For $\delta > \gamma$ we must take into account the pole $z = 0$, where the integrand has a residue

$$Q_y = \frac{(-1)^k y^s}{s^k (k-1)!} \sum_{m=0}^{k-1} \frac{(-1)^m (k+m-1)!}{m! s^m} \sum_{r=m+1}^{k} a_{-r,k} \frac{\log^{r-m-1} x}{(r-m-1)!},$$

so that altogether

$$J_1 = S_y - \varepsilon_\gamma Q_y,$$

where $\varepsilon_\gamma = 0$ if $\delta < \gamma$ and $\varepsilon_\gamma = 1$ if $\delta > \gamma$.

The line of integration in J_2 is moved to $\mathrm{Re}\, z = 1/4$, and for $\delta < \gamma$ the pole $z = 0$ of the integrand is passed. In calculating the residue note that $X(0) = X'(0) = \dots = X^{(k-1)}(0) = 0$, since in $X(u)$ and its first k-1 derivatives the factor $\sin(\pi u/2)$ comes in. This leads to

$$J_2 = (2\pi i)^{-1} \int_{1/4 - i\infty}^{1/4 + i\infty} \zeta^k(1-z)\phi(z)(z-s)^{-k} y^{s-z} dz - (1-\varepsilon_\gamma) X(s) Q_y = \tag{4.37}$$

$$J_y - (1 - \varepsilon_\gamma) X(s) Q_y.$$

Inserting the expressions (4.36) and (4.37) in (4.35) we obtain

$$F_x - S_x + Q_x = -X(s)J_1 - J_2 = -X(s)(S_y - Q_y) - J_y. \tag{4.38}$$

At this stage of the proof a Tauberian argument comes into play. The underlying idea is that (4.38) remains true if x and y are replaced by $xe^{\nu h}$ and $xe^{-\nu h}$ respectively, where $0 < h \le 1$ and ν is an integer for which $\nu \le k-1$, and moreover h will be suitably chosen later. Now we shall sum (4.38) with weight $(-1)^\nu \binom{k-1}{\nu}$ for $0 \le \nu \le k-1$ to recover the approximate functional equation by means of the elementary identity

$$\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} \nu^p = \begin{cases} 0 & p < m \\ m! & p = m \end{cases}, \tag{4.39}$$

and the estimate

$$e^z = \sum_{n=0}^{m-1} z^n/n! + O(|z|^m), \quad m \ge 2, \quad a \le \mathrm{Re}\, z \le b, \tag{4.40}$$

where a and b are fixed.

To distinguish better the sums which will arise in this process we introduce left indexes to obtain then from (4.38)

$$\sum_{\nu=0}^{k-1}(-1)^{\nu}\binom{k-1}{\nu}\left({}_{\nu}F_x - {}_{\nu}S_x + {}_{\nu}Q_x + X(s)\,{}_{\nu}S_y - X(s)\,{}_{\nu}Q_y + {}_{\nu}J_y\right) = 0,$$

or abbreviating,

$$\bar{F}_x - \bar{S}_x + \bar{Q}_x + X(s)\bar{S}_y - X(s)\bar{Q}_x + \bar{J}_y = 0. \tag{4.41}$$

Each term in (4.41) will be evaluated now separately. We have

$$\bar{F}_x = \sum_{m=0}^{k-1}\frac{(\zeta^k(s))^{(m)}}{m!\,(k-1-m)!}A_m(x),$$

where we have set

$$A_m(x) = \sum_{\nu=0}^{k-1}(-1)^{\nu}\binom{k-1}{\nu}(\log x + \nu h)^{k-1-m}.$$

Using (4.39) it follows that

$$A_m(x) = \sum_{r=0}^{k-1}\binom{k-1-m}{r}h^r\log^{k-1-m-r}x\sum_{\nu=0}^{k-1}(-1)^{\nu}\binom{k-1}{\nu}\nu^r = h^{k-1}(k-1)!$$

when $m = 0$ and $A_m(x) = 0$ when $m > 0$, so that we have

$$\bar{F}_x = h^{k-1}\zeta^k(s), \tag{4.42}$$

and this is exactly what is needed for the approximate functional equation that will eventually follow on dividing (4.41) by h^{k-1} with a suitably chosen h. Consider next

$$\bar{S}_x = \frac{1}{(k-1)!}\sum_{n\le x}d_k(n)n^{-s}\sum_{\nu=0}^{k-1}\binom{k-1}{\nu}(-1)^{\nu}(\nu h + \log x/n)^{k-1} +$$

$$+ \frac{1}{(k-1)!}\sum_{\nu=0}^{k-1}(-1)^{\nu}\binom{k-1}{\nu}\sum_{x<n\le xe^{\nu h}}d_k(n)n^{-s}(\nu h + \log x/n)^{k-1} = \sum\nolimits_1 + \sum\nolimits_2, \tag{4.43}$$

say. Analogously to the evaluation of $\bar{F}_x$ it follows on using (4.39) again

$$\sum\nolimits_1 = h^{k-1}\sum_{n\le x}d_k(n)n^{-s}, \tag{4.44}$$

and we estimate $\sum_2$ trivially (using $d_k(n) \ll n^{\varepsilon}$) as

$$\left|\sum\nolimits_2\right| \leq \frac{1}{(k-1)!}\sum_{\nu=0}^{k-1}\binom{k-1}{\nu}(\nu h)^{k-1}x^{-\delta}\sum_{x<n\leq xe^{(k-1)h}} d_k(n) \ll$$

(4.45)

$$h^{k-1}x^{-\delta}t^{\varepsilon}(1 + x(e^{(k-1)h} - 1)) \ll t^{\varepsilon}(h^{k-1}x^{-\delta} + h^k x^{1-\delta}).$$

Estimating analogously $\overline{S}_y$ we obtain

$$(4.46) \quad -X(s)\overline{S}_y = h^{k-1}\chi^k(s)\sum_{n\leq y} d_k(n)n^{s-1} + O(h^{k-1}t^{\varepsilon}x^{1/2-\delta}y^{-1/2}) + O(h^k t^{\varepsilon}x^{1/2-\delta}y^{1/2}).$$

Next we have

$$\overline{Q}_x = \frac{x^{1-s}}{(k-1)!(1-s)^k}\sum_{\mu=0}^{k-1}\frac{(-1)^{\mu}(k+\mu-1)!}{\mu!(1-s)^{\mu}}\sum_{\rho=\mu+1}^{k}\frac{a_{-\rho,k}}{(\rho-\mu-1)!}B_{\mu\rho},$$

where

$$B_{\mu\rho} = \sum_{\nu=0}^{k-1}(-1)^{\nu}\binom{k-1}{\nu}e^{\nu h(1-s)}(\nu h + \log x)^{\rho-\mu-1}.$$

Using (4.39) and (4.40) with $m = k$ it follows

$$B_{\mu\rho} = (k-1)!h^{k-1}\sum_{\substack{n+m=k-1\\ 0<m<\rho-m-1}}(n!)^{-1}(1-s)^n\log^{\rho-\mu-1-m}x\binom{\rho-\mu-1}{m} + O(h^k t^k\log^{\rho-\mu-1}x).$$

If we set $\nu = \rho-\mu-1-m$, change the order of summation and collect the constants we obtain

$$(4.47) \qquad \overline{Q}_x = x^{1-s}(1-s)^{-1}h^{k-1}\sum_{\nu=0}^{k-1}\sum_{\rho=\nu+1}^{k} a_{-\rho,k}r_{\rho,\nu}(1-s)^{1-\nu-\rho}\log^{\nu}x + O(h^k x^{1+\varepsilon-\delta}) =$$

$$h^{k-1}R_x(s) + O(h^k x^{1+\varepsilon-\delta}).$$

The same argument applies to $\overline{Q}_y$ and yields

$$(4.48) \qquad \overline{Q}_y = -h^{k-1}R_y(1-s) + O(h^k x^{1/2+\varepsilon-\delta}y^{1/2}).$$

Therefore we are left with the evaluation of

$$\overline{J}_y = (2\pi i)^{-1}\int_{1/4-i\infty}^{1/4+i\infty}\zeta^k(1-z)\phi(z)y^{s-z}(z-s)^{-k}\left(\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}e^{-hm(s-z)}\right)dz.$$

Observing that $\phi(z)$ has a double zero at $z = s$ and that

$$\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}e^{-hm(s-z)} = (1-e^{-hs+hz})^{k-1}$$

has a zero of order k-1 at z = s, we can move the line of integration in $\bar{J}_y$ to Re $z = 1/2$ to obtain with $w = u + iv$ (u,v real)

$$(4.49)\quad \bar{J}_y = (2\pi i)^{-1}\int_{1/2-i\infty}^{1/2+i\infty}\zeta^k(1-w)\phi(w)(w-s)^{-k}y^{s-w}\Big(\sum_{m=0}^{k-1}(-1)^m\binom{k-1}{m}e^{-hm(s-w)}\Big)dw =$$

$$(2\pi i)^{-1}\Big(\int_{|v-t|\le G}\ldots dw + \int_{|v-t|>G,\,|v|\le 2t^{\beta}}\ldots dw + \int_{|v|\ge 2t^{\beta}}\ldots dw\Big) = j_1 + j_2 + j_3,$$

say. Here β (≥ 1) is the number appearing in (4.27), and $t^{\varepsilon} < G \le t^{2/3}$ is a parameter that will be suitably chosen. We distinguish now the cases $3 \le k \le 4$ and $k > 4$, and treat first the latter case. For j_1 we use (4.31) in the form

$$\phi(w) \ll t^{k/2-k\delta-1}|s-w|^2,$$

and majorize the sum $\sum_{m=0}^{k-1}$ in (4.49) by $O(h^{k-1}|s-w|^{k-1})$, which follows when we combine (4.39) and (4.40). Therefore we have

$$(4.50)\quad j_1 \ll h^{k-1}y^{\delta-1/2}t^{k(1/2-\delta)-1}G\int_{t-G}^{t+G}|\zeta(1/2+iv)|^k dv \ll$$

$$h^{k-1}x^{1/2-\delta}Gt^{-1}\int_{t-t^{2/3}}^{t+t^{2/3}}|\zeta(1/2+iv)|^k dv \ll h^{k-1}x^{1/2-\delta}Gt^{(k-4)c+\varepsilon-1/3},$$

where (4.32) was used. To estimate j_2 we use $\phi(w) \ll t^{k/2-k\delta}$ and the same majorization for $\sum_{m=0}^{k-1}$ as above to obtain

$$(4.51)\quad j_2 \ll h^{k-1}x^{1/2-\delta}\int_{|v-t|>G,\,|v|\le 2t^{\beta}}|\zeta(1/2+iv)|^k|v-t|^{-1}dv.$$

The integral in (4.51) is split into subintegrals $j_{21}, j_{22}, j_{23}, j_{24}, j_{25}$ over the intervals $[-2t^{\beta},-2t]$, $[-2t,t/2]$, $[t/2,t-G]$, $[t+G,2t]$, $[2t,2t^{\beta}]$ respectively. Using (4.33) it follows at once that

$$j_{22} \ll t^{(k-4)/8+\varepsilon},$$

and the other integrals are integrated by parts and then estimated. For example for

j_{23} we have with

$$H(v) = -\int_v^{t-G} |\zeta(1/2 + ix)|^k dx$$

and (4.33) that

(4.52) $$j_{23} = H(v)(t-v)^{-1}\Big|_{t/2}^{t-G} - \int_{t/2}^{t-G} H(v)(t-v)^{-2}dv \ll$$

$$t^{(k-4)/8+\varepsilon} + G^{-1}t^{1+(k-4)/8+\varepsilon} \ll G^{-1}t^{1+(k-4)/8+\varepsilon},$$

since $G \le t^{2/3}$, and the same bound similarly holds for j_{24}. Using again (4.33) we have

$$j_{21} + j_{25} \ll t^{\beta((k-4)/8+\varepsilon)},$$

and so

(4.53) $$j_1 + j_2 \ll t^{\varepsilon}(h^{k-1}x^{1/2-\delta}Gt^{(k-4)c-1/3} + h^{k-1}x^{1/2-\delta}G^{-1}t^{1+(k-4)/8} + $$

$$+ h^{k-1}x^{1/2-\delta}t^{\beta(k-4)/8}).$$

We choose now G in such a way that the first two terms on the right-hand side of (4.53) are equal. Thus with $c = 35/216$ we let

$$G = t^{2/3+(k-4)(1/8-c)/2}.$$

This choice of G obviously satisfies the condition $t^{\varepsilon} < G \le t^{2/3}$, and then we obtain

(4.54) $$j_1 + j_2 \ll t^{\varepsilon}h^{k-1}x^{1/2-\delta}(t^{(31k-52)/216} + t^{\beta(k-4)/8}) \ll$$

$$t^{\varepsilon}h^{k-1}x^{1/2-\delta}t^{(31k-52)/216},$$

if as in (4.27) we take

(4.55) $$\beta = (31k - 52)/(27k - 108) > 1.$$

Integration by parts and (4.33) give

(4.56) $$j_3 \ll x^{1/2-\delta}\int_{2t^{\beta}}^{\infty}|\zeta(1/2 + iv)|^k v^{-k}dv \ll x^{1/2-\delta}t^{\beta(4-7k+\varepsilon)/8},$$

where we used $\phi(w) \ll t^{k/2-k\delta}$ and $\sum_{m=0}^{k-1} \ll 1$ for the sum appearing in (4.49), since $e^{-hm(s-w)} = e^{-hm(\delta-1/2)} \ll 1$. Finally combining all the estimates (4.41) (4.56) we obtain

$$h^{k-1}\Big(\zeta^k(s) - \sum_{n\le x} d_k(n)n^{-s} - \chi^k(s)\sum_{n\le y} d_k(n)n^{s-1} + R_x(s) + \chi^k(s)R_y(1-s)\Big) \ll \tag{4.57}$$

$$t^{\varepsilon}\big(h^k(x+y)^{1/2}x^{1/2-\delta} + h^{k-1}x^{1/2-\delta}t^{(31k-52)/216} + x^{1/2-\delta}t^{\beta(4-7k)/8}\big).$$

Choosing $h = t^{-\beta}$, where β is already defined by (4.55), it is seen that the last two terms in (4.57) are equal, and the approximate functional equation follows from (4.57) on dividing by h^{k-1}, if we recall that the error made by replacing the condition $xy = (t/2\pi)^k$ by $xy = T$ is $\ll x^{1/2-\delta}\min(x^{1/2},y^{1/2})t^{\varepsilon-2}$.

This settles the case $k > 4$, and we have still to consider the cases $k = 3$ and $k = 4$. The only changes in the proof will be in the estimation of the integrals j_1, j_2, j_3 appearing in (4.49), where sharper estimates than those used for the general case $k > 4$ are available.

For $k = 3$ we choose $G = 2t^{1/2}$ in (4.49) to obtain with the third moment estimate (6.75)

$$j_1 \ll h^2x^{1/2-\delta}t^{-1/2}\int_{t-2t^{1/2}}^{t+2t^{1/2}} |\zeta(1/2 + iv)|^3dv \ll h^2x^{1/2-\delta}t^{1/8+\varepsilon}.$$

For j_{23} we use (6.75) again to obtain

$$j_{23} \ll h^2x^{1/2-\delta}\int_{t/2}^{t-2t^{1/2}} |\zeta(1/2 + iv)|^3(t-v)^{-1}dv \ll$$

$$h^2x^{1/2-\delta}\sum_{n=1}^{O(t^{1/2})}\int_{t-2(n+1)t^{1/2}}^{t-2nt^{1/2}} |\zeta(1/2 + iv)|^3(t-v)^{-1}dv \ll$$

$$h^2x^{1/2-\delta}\sum_{n=1}^{O(t^{1/2})}t^{-1/2}n^{-1}\int_{t-2nt^{1/2}-2t^{1/2}}^{t-2nt^{1/2}} |\zeta(1/2 + iv)|^3dv \ll$$

$$h^2x^{1/2-\delta}t^{\varepsilon-1/2}\sum_{n=1}^{O(t^{1/2})}n^{-1}(t^{1/2} + t^{5/8}) \ll h^2x^{1/2-\delta}t^{1/8+\varepsilon}.$$

At last using $\int_0^T|\zeta(1/2+it)|^3dt \ll T^{1+\varepsilon}$ we obtain

$$j_1 + j_2 + j_3 \ll t^{\varepsilon}x^{1/2-\delta}(h^2t^{1/8} + t^{-2\beta}).$$

Thus for $k = 3$ we obtain (4.57) where the right-hand side will be

$$\ll t^{\varepsilon}(h^3(x + y)^{1/2}x^{1/2-\delta} + h^2x^{1/2-\delta}t^{1/8} + x^{1/2-\delta}t^{-2\beta}).$$

Now we set $\beta = 11/8$, $h = t^{-\beta}$. Then from $x + y \ll t^3$ we infer that

$$h^3(x + y)^{1/2}x^{1/2-\delta} \ll h^2x^{1/2-\delta}t^{1/8},$$

and dividing (4.57) by h^2 we obtain the desired approximate functional equation

$$(4.58) \quad \zeta^3(s) = \sum_{n\le x} d_3(n)n^{-s} + \chi^3(s)\sum_{n\le y} d_3(n)n^{s-1} - R_x(s) - \chi^3(s)R_y(1-s) + O(x^{1/2-\delta}t^{\frac{4}{3}+\epsilon}).$$

It remains to consider yet the case $k = 4$, where the estimation is identical with the general case up to (4.51), only now for $H(v)$ we shall use

$$\int_0^T |\zeta(1/2 + it)|^4 dt = T\sum_{j=0}^{4} a_j \log^{4-j}T + O(T^{7/8+\epsilon}),$$

which is a result of D.R. Heath-Brown [3] (see Notes of Chapter 5). Therefore for j_{23} in (4.51) with $k = 4$ we have

$$j_{23} \ll t^{\epsilon}\Big(1 + \int_{t/2}^{t-G} (t - v - G + t^{7/8})(t - v)^{-2}dv\Big) \ll G^{-1}t^{7/8+\epsilon},$$

which means that we have saved a factor $t^{1/8}$ from the general estimate used in (4.51). We have then

$$j_1 + j_2 \ll t^{\epsilon}(h^3x^{1/2-\delta}Gt^{-4/3} + h^3x^{1/2-\delta}G^{-1}t^{7/8}) \ll h^3x^{1/2-\delta}t^{13/48+\epsilon}$$

for $G = t^{29/48} < t^{2/3}$. Since $j_3 \ll x^{1/2-\delta}t^{\beta(\epsilon-3)}$ we obtain (4.57) where the right-hand side will be

$$\ll t^{\epsilon}(h^4(x + y)^{1/2}x^{1/2-\delta} + h^3x^{1/2-\delta}t^{13/48+\epsilon} + x^{1/2-\delta}t^{-3\beta}).$$

The result given by (4.26) follows for $\beta = 2$, $h = t^{-\beta}$ on dividing (4.57) by h^3, since

$$h(x + y)^{1/2}x^{1/2-\delta} \ll t^{-2}(t^4)^{1/2}x^{1/2-\delta} = x^{1/2-\delta}.$$

§4. The reflection principle

The approximate functional equations discussed in §2 and §3 have a symmetric property if $x = y$, especially when $\delta = 1/2$, which allows one to obtain useful expressions like (4.12), (4.13), (4.28) and (4.29). However, when one seeks estimates for averages of powers of moduli of the zeta-function in the critical strip, it turns out that the approximate functional equations of the type just considered have two shortcomings. Firstly the lengths of the sums over n depend on t, and secondly the error terms for $k \geq 3$ and $\sigma = 1/2$ already are not small (i.e. they are not $\ll t^{\varepsilon}$). We shall proceed now to derive another type of approximate functional equation, which though lacking symmetry is in many problems concerning averages of the zeta-function quite adequate. The idea, which permeates the whole theory since the pioneering days of Riemann, is to use the functional equation in the form $\zeta^k(w) = \chi^k(w)\zeta^k(1-w)$ for some w with $\mathrm{Re}\, w < 0$, to split the series $\zeta^k(1-w) = \sum_{n=1}^{\infty} d_k(n)n^{w-1}$ at some suitable M and estimate the terms with $n > M$ trivially. This approach is very flexible, and the error terms that will arise will be small. The starting point is the Mellin integral (1.7) where we set $x = Y^h$, $s = w/h$ and suppose $Y, h > 0$. In view of $\Gamma(z+1) = z\Gamma(z)$ we obtain by moving the line of integration

$$e^{-Y^h} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} Y^{-w}\Gamma(1 + w/h)w^{-1}dw. \tag{4.59}$$

Replacing now Y by n/Y and using $\sum_{n=1}^{\infty} d_k(n)n^{-z} = \zeta^k(z)$ ($\mathrm{Re}\, z > 1$) it follows when $\sigma \geq 0$ and $k \geq 1$ is an integer that

$$\sum_{n=1}^{\infty} e^{-(n/Y)^h} d_k(n)n^{-s} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} \zeta^k(s + w)Y^w\Gamma(1 + w/h)w^{-1}dw. \tag{4.60}$$

Now we suppose $s = \sigma + it$, $0 \leq \sigma \leq 1$, $h^2 \leq t \leq T$, $h = \log^2 T$, $1 \ll Y \ll T^c$ for some fixed $c > 0$, and we move the line of integration in (4.60) to $\mathrm{Re}(s+w) = -1/2$. Using Stirling's formula (1.32) it is seen that the residue coming from the pole $w = 1 - s$ is $o(1)$, while the residue at the pole $w = 0$ is $\zeta^k(s)$. Using the functional equation (4.3) we have then

$$\sum_{n=1}^{\infty} d_k(n)e^{-(n/Y)^h}n^{-s} = \zeta^k(s) + o(1) + I_1 + I_2, \tag{4.61}$$

say, where for some $1 \ll M \ll T^c$

$$I_1 = (2\pi i)^{-1} \int_{\mathrm{Re}(s+w)=-1/2} \chi^k(s+w)\sum_{n \leq M} d_k(n)n^{w+s-1}Y^w\Gamma(1 + w/h)w^{-1}dw, \tag{4.62}$$

$$I_2 = (2\pi i)^{-1} \int_{\mathrm{Re}(s+w)=-1/2} \chi^k(s+w)\sum_{n > M} d_k(n)n^{w+s-1}Y^w\Gamma(1 + w/h)w^{-1}dw. \tag{4.63}$$

In I_2 we move the line of integration to $\mathrm{Re}(s+w) = -h/2$, noting that the integrand is regular for $-h/2 \le \mathrm{Re}(s+w) \le -1/2$, and aiming to choose M in such a way that $I_2 = o(1)$ as $T \to \infty$. With $w = u + iv$ (u,v real) we obtain using Stirling's formula

$$I_2 \ll \int_{-\infty}^{\infty} |\chi(-\tfrac{h}{2} + iv + it)|^k \sum_{n>M} d_k(n) n^{-1-h/2} Y^{-h/2} |\Gamma(1/2 - \tfrac{\alpha}{h} + \tfrac{i}{h}v)| dv \ll$$

$$(MY)^{-h/2} \log^k T \int_0^T (t+v)^{k(1+h)/2} dv + \int_T^{\infty} e^{-v/h} dv \ll$$

$$(MY)^{-h/2} T \log^k T \cdot (2T)^{k(1+h)/2} + o(1) = o(1)$$

if

$$M \ge (3T)^k Y^{-1}. \tag{4.64}$$

The flexibility of this method is best seen in various possibilities for the estimation of I_1 in (4.62). The line of integration in I_1 may be moved to $\mathrm{Re}(s+w) = \alpha$, $0 < \alpha < 1$ fixed and $\alpha \ne \sigma$, so that the sum appearing in (4.62) will be "reflected", hence the name "reflection principle". Letting $\delta_\alpha = 1$ if $\alpha > \sigma$ and $\delta_\alpha = 0$ if $\alpha < \sigma$ we obtain by the residue theorem

$$I_1 = -\delta_\alpha \chi^k(s) \sum_{n \le M} d_k(n) n^{s-1} + (2\pi i)^{-1} \int_{\mathrm{Re}(s+w)=\alpha} \chi^k(s+w) \sum_{n \le M} d_k(n) n^{w+s-1} Y^w \Gamma(1 + w/h) \frac{dw}{w}. \tag{4.65}$$

The terms with $n > 2Y$ in (4.61) are trivially $o(1)$, and the part of the integral in (4.65) with $|v| = |\mathrm{Im}\, w| \ge h^2$ is also $o(1)$ by Stirling's formula, so that combining (4.61)-(4.65) we have

$$\zeta^k(s) = \sum_{n \le 2Y} d_k(n) e^{-(n/Y)^h} n^{-s} + \delta_\alpha \chi^k(s) \sum_{n \le M} d_k(n) n^{s-1} -$$
$$- (2\pi i)^{-1} \int_{\substack{\mathrm{Re}(s+w)=\alpha \\ |\mathrm{Im}\, w| \le h^2}} \chi^k(s+w) \sum_{n \le M} d_k(n) n^{w+s-1} Y^w \Gamma(1 + w/h) w^{-1} dw + o(1). \tag{4.66}$$

Therefore we have obtained the desired type of the (unsymmetrical) approximate functional equation, where the lengths of the sums do not depend on t, but on T, and where the error term is $o(1)$. Another useful variant follows from (4.66) with $k = 1, \alpha = 1/2$ when we replace Y by 2Y and subtract the resulting expressions, or proceed directly and observe that there is now no pole at $w = 0$ because of the zero $w = 0$ of $(2Y)^w - Y^w$. We obtain then

$$(4.67) \qquad \sum_{n=1}^{\infty}(e^{-(n/2Y)^h} - e^{-(n/Y)^h})n^{-s} \ll 1 + Y^{1/2-\delta}\int_{-h^2}^{h^2}\Big|\sum_{n\le M} n^{-1/2+it+iv}\Big|dv,$$

which will be very useful later in Chapter 9 for zero-density estimates. In (4.67) we have $0 \le \delta \le 1$, $h^2 \le t \le T$, $1 \ll Y \ll T^c$, $M \ge 3TY^{-1}$, $h = \log^2 T$.

NOTES

B. Riemann's classical memoir [1], his only work from analytic number theory, is extensively discussed by H.M. Edwards [1].

For many different and instructive proofs of the functional equation (4.1) the reader should consult Chapter 2 of Titchmarsh [8]. The proof of the functional equation presented in §1 is due to J. van de Lune [1].

One may prove (4.8) with $f(x)$ given by (4.7) as follows. Substituting $x = -\frac{i}{2}u$ in the identity $\sin x = x\prod_{k=1}^{\infty}(1 - x^2/(\pi^2k^2))$ one obtains by logarithmic differentiation

$$\frac{1}{e^u - 1} = \frac{1}{u} - \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2u}{4k^2\pi^2 + u^2}. \qquad (u \ne 2n\pi i)$$

With $f(x)$ defined by (4.7) we have then

$$\int_0^{\infty} f(x)\sin xt\cdot dx = \int_0^{\infty}((e^{\sqrt{2\pi}x} - 1)^{-1} - (\sqrt{2\pi}x)^{-1})\sin xt\cdot dx =$$

$$- (2\pi)^{-1/2}\int_0^{\infty} x^{-1}\sin xt\cdot dx + \sum_{k=1}^{\infty}\int_0^{\infty} e^{-kx\sqrt{2\pi}}\sin xt\cdot dx =$$

$$- \frac{1}{4}\sqrt{2\pi} + \sum_{k=1}^{\infty}\frac{t}{2\pi k^2 + t^2} = - \frac{1}{4}\sqrt{2\pi} + \frac{1}{2}\sqrt{2\pi}\Big(\frac{1}{e^{\sqrt{2\pi}t} - 1} - (\sqrt{2\pi}t)^{-1} + 1/2\Big) =$$

$$= \sqrt{\pi/2}f(t),$$

which gives then (4.8) and completes our discussion of the functional equation.

As was also the case with Chapter 3, none of the results of this chapter are explicitly formulated as theorems. This was done on purpose to emphasize the flexibility of the approach, especially in our discussion of the reflection principle. Also some of the results, like (4.1), (4.10) and (4.11) should be already known to the reader.

There is a proof given by Titchmarsh [8] of (4.10) by what amounts to the use of a variant of Poisson's summation formula (Lemma 2.4), only the error term $O(x^{-\sigma}\log t)$ instead of $O(x^{-\sigma})$ is obtained. There is also a proof of (4.10) in full strength given by Titchmarsh [8] by a contour integration method, while (4.11) is stated at the end of Chapter 4 with the error term $O((x+y)^{1/2-\sigma}\log t)$ without proof. It has been kindly pointed out to me by M. Jutila that this error term must be incorrect, since $f(x) = \sum_{n\leq x} d(n)n^{-s}$ is discontinuous at integers with jumps $d(x)x^{-s}$, accordingly the error term should be at least $O(x^{-\sigma})$. But if x is small and σ is near unity, then $(x + y)^{1/2-\sigma}\log t$ is much smaller than $x^{-\sigma}$. The correct error term in (4.11), which is $O(x^{1/2-\sigma}\log t)$ is obtained by Titchmarsh [5]. His method of proof there is an extension of the proof used by Hardy and Littlewood [3] in the proof of the approximate functional equation (4.10) for $\zeta(s)$. The first step is to obtain an exact formula for $\zeta^2(s)$, valid for $\sigma > -1/4$. This is

$$\zeta^2(s) = \sum_{n\leq x} d(n)n^{-s} - x^{-s}\sum_{n\leq x} d(n) + \frac{2s-s^2}{(s-1)^2}x^{1-s} + \frac{s}{s-1}x^{1-s}(2\gamma + \log x) +$$

$$+ \frac{1}{4}x^{-s} - 2^{4s}\pi^{2s-2}s\sum_{n=1}^{\infty} d(n)n^{s-1}\int_{4\pi\sqrt{nx}}^{\infty} (K_1(v) + \tfrac{1}{2}\pi Y_1(v))v^{-2s}dv,$$

and then using the asymptotic expansions for the Bessel functions K_1, Y_1 (see Chapter 3) one is led to the estimation of certain exponential integrals which eventually yield (4.11).

The derivation of the approximate functional equation (4.11) for $\zeta^2(s)$ in §2 is novel and illustrates well the power of the Voronoï summation formula. However the possibility of such an approach has been mentioned by M. Jutila [6], whose idea to exploit the Voronoï summation formula is used here. In the proof of (4.11) it may be assumed without loss of generality that $x \geq y$, for on dividing by $\chi^2(s)$ ($\asymp t^{1-2\sigma}$) and then changing s into 1 - s, one obtains the corresponding result with x and y interchanged.

Note that the definition of $\Delta(x)$ given in §2 slightly differs from the one made in (3.1), but this can cause no confusion since the two expressions differ only by $O(x^{\varepsilon})$, which is absorbed in (4.15) in the error term $O(x^{1/2-\sigma}\log t)$.

To investigate more precisely the absolute convergence of the integral in (4.14) we proceed as follows. By Theorem 10.5 and the Cauchy-Schwarz inequality for integrals we have

$$\left|\int_M^{2M} x^{-s-1}\Delta(x)dx\right| \le \left(\int_M^{2M} x^{-2\sigma-2}dx\right)^{1/2}\left(\int_M^{2M}\Delta^2(x)dx\right)^{1/2} \ll (M^{-1-2\sigma}M^{3/2})^{1/2} \ll M^{-\varepsilon}$$

for $\sigma > 1/4$. Taking $M = N, 2N, 2^2N, \dots$ etc. and adding up the estimates it is seen that the integral in (4.14) is absolutely convergent for $\sigma > 1/4$.

To see how the error term $O(x^{1/2-\sigma}\log t)$ appears in (4.15) we use (3.1) in evaluating $\int_x^N u^{-s}d\Delta(u)$. Writing

$$\Delta(x) = \sum_{n\le(1+\varepsilon)y} + \sum_{n>(1+\varepsilon)y}$$

the first sum in view of (3.1) and (3.11) gives the first sum with K_0 and Y_0 on the right-hand side of (4.15). Integration by parts gives

$$\int_x^N u^{-s}d\Big(\sum_{n>(1+\varepsilon)y}\dots\Big) = \Big(\sum_{n>(1+\varepsilon)y}\dots\Big)u^{-s}\Big|_x^N + s\sum_{n>(1+\varepsilon)y} d(n)\int_x^N(-2\pi(u/n)^{1/2}(K_1(\dots))du,$$

while (3.16) gives

$$x^{-s}\sum_{n>(1+\varepsilon)y}\{-2(x/n)^{1/2}(K_1(4\pi\sqrt{nx}) + \tfrac{\pi}{2}Y_1(4\pi\sqrt{nx})\} \ll$$

$$x^{-\sigma}(|\Delta(x)| + (x/y)^{1/2}x^{\varepsilon}) \ll x^{1/2-\sigma}\log t,$$

since $y \gg t^{\varepsilon}$.

The error terms in the approximate functional equation for $\zeta^{k}(s)$ in §3 are due to the author, and this result has not appeared in print before. The method of proof is based on R. Wiebelitz [1], who was guided by the work of Hardy and Littlewood [3]. Estimates (6.71) and (6.75) are used in the proof, as is also Heath-Brown's fourth power moment estimate [3]. The proofs of these results fall beyond the scope of this text, and the results of §3 are among the few ones whose proof is not self-contained. For comparison we present now Wiebelitz's approximate functional equation for $\zeta^k(s)$, so that improvements obtained in §3 may be seen.

Wiebelitz uses the estimate $\zeta(1/2+it) \ll t^{c+\varepsilon}$, $c = 15/92$ (due to S.-H. Min [1]), which was the best result available at the time of his writing, and supposes $k \ge 3$ is a fixed integer, $xy = (|t|/2\pi)^k$; $x,y \gg 1$, $-1/2 \le \sigma \le 3/2$. Then (4.25) holds uniformly in σ with

$$\Delta_k(x,y) \ll (x+y)^{1/2}x^{1/2-\delta}|t|^{-\beta}\log^{k-1}|t| + x^{1/2-\delta}(x+y)^{2\beta c(k-4)/k+\varepsilon} +$$

$$+ x^{1/2-\delta}|t|^{(k-2)c}\log^2|t| + x^{1/2-\delta}\min(x^{1/2},y^{1/2})|t|^{-2},$$

where $\beta = 3/2$ for $k = 3$ and $\beta = 23k/(15k+32)$ for $k \geq 4$. The terms of R_x and R_y in (4.25) for $\rho \geq \beta + \gamma$ may be incorporated in the above error terms, which was done by Wiebelitz. Though Wiebelitz proves his result for $-1/2 \leq \delta \leq 3/2$ (curiously, it seems to be his only paper in number theory), the result is really of interest for $0 \leq \delta \leq 1$ (and especially for $\delta \geq 1/2$) in view of the functional equation (4.1).

When estimating $\sum_2$ in (4.45) Wiebelitz uses the asymptotic formula for $\sum_{n\leq x} d_k(n)$, which will be extensively discussed in Chapter 10. This enabled him to have $h^{k-1}x^{-\delta}\log^{k-1}|t|$, while in (4.45) we had $h^{k-1}x^{-\delta}t^{\varepsilon}$, but introduces the error term $\Delta_k(x)$ in the general divisor problem, which ultimately affects Wiebelitz's estimate for $\Delta_k(x,y)$, as given above. Using the trivial estimate

$$\sum_{a<n\leq b} d_k(n) \ll b^{\varepsilon}(1+b-a)$$

in (4.45) we managed to dispose of the error term $\Delta_k(x)$, and as the estimates for power moments of the zeta-function that were used involve the factor t^{ε}, we would gain nothing by following Wiebelitz in the use of $\Delta_k(x)$ in (4.45). Estimates given in §3 for $\Delta_k(x,y)$ are clearly superior to the corresponding ones given by Wiebelitz.

Concerning (4.30) observe that by Taylor's formula

$$T^{u-s}X(u) = X(s) + (u-s)T^{u-s}(X'(s) + X(s)\log T) + \frac{1}{2!}(X''(s)T^{u-s} + \ldots)(u-s)^2 + \ldots,$$

and since by (4.20) we have

$$X'(1/2+it) + X(1/2+it)\log T = 0,$$

it is seen that $\phi_1(u)$ is regular for $\mathrm{Re}\,u = 1/2$, when the double zeros $u = s = \frac{1}{2} + it$ of the numerator and denominator cancel each other, and the other ranges for u are easy. This discussion also shows why the definition of T in (4.20), which may have looked a little mysterious, is a natural one to make.

To see that (4.31) holds observe that for $\mathrm{Re}\,u = 1/2$ we have $|X(u)| = 1$, so that by (4.4)

$$|\phi(u)| \leq |T^{u-s}| + |X(s)| \ll t^{k(1/2-\delta)}.$$

This proves the first estimate in (4.31), and for the second one note that if $|s-u|^2 \le t$, then $\mathrm{Re}\, u \asymp t$, and since we have $\frac{d^2}{ds^2}(\log\chi(s)) \ll t^{-1}$, we may write

$$\phi(u) = T^{u-s}X(u)(1 - \frac{X(s)}{X(u)}T^{s-u})$$

and use Taylor's formula and (4.20) with $u = 1/2 + iv$, v real. This gives

$$1 - \frac{X(s)}{X(u)}T^{s-u} = 1 - \exp(-k(s-u)\frac{\chi'}{\chi}(1/2+it)+k\log\chi(\delta+it)-k\log\chi(1/2+iv)) =$$

$$1 - \exp(-k(s-u)\left\{\frac{\chi'}{\chi}(1/2+it) - \frac{\chi'}{\chi}(1/2+iv)\right\} + O(|s-u|^2t^{-1})) =$$

$$1 - \exp(k(\delta + it - iv - 1/2)^2 O(t^{-1})) \ll |s - u|^2 t^{-1},$$

and then (4.31) follows.

To prove (4.39) one may start from

$$(1 + x)^m = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{m}x^m$$

and differentiate, taking eventually $x = -1$. In the first step we have

$$m(1 + x)^{m-1} = \binom{m}{1} + 2\binom{m}{2}x + \dots + m\binom{m}{m}x^{m-1},$$

and the proof is finished if $p = 1$ by taking $x = -1$. If $p \neq 1$, then the above equation is multiplied by x and differentiated again and the process is repeated sufficiently many times. Finally for $p = m$ we obtain

$$\sum_{\nu=0}^{m}(-1)^\nu \binom{m}{\nu}\nu^p = m!,$$

since we arrive at an expression whose left side is $m!$ plus a polynomial in $x + 1$, and taking $x = -1$ we have the above identity.

In (4.35) one uses $T^{u-s}X(u) = X(s) + \phi(u)$, which is the main reason why $\phi(u)$ was introduced by (4.30).

The discussion of the reflection principle in §4 is based mostly on M. Jutila [2]. This simple and powerful method was used in a similar form before Jutila's work by M.N. Huxley [3] and K. Ramachandra [1]. A general principle in analytic number theory is to express a sum (or series) by a contour integral in the complex plane, and to attain flexibility by moving the contour of integration and applying the residue theorem. When coupled with the use of the functional (equation) (4.1), this idea leads to the reflection principle.

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TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 5

THE FOURTH POWER MOMENT

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§1. Introduction

Estimates of integrals of the type $\int_0^T |\zeta(1/2+it)|^k dt$ play a prominent role in many parts of the zeta-function theory, and applications of these estimates to zero-density theorems will be given in Chapter 9. The important case when $k = 2$ is one of the main topics of this text, and will be extensively treated in Chapter 11. A detailed account of Atkinson's formula for $E(T)$ and some of its applications will be presented there, while power moment estimates for $k > 4$ will be treated in Chapter 7. This chapter is devoted to the case $k = 4$, and we shall prove the following

THEOREM 5.1.

$$(5.1) \qquad \int_0^T |\zeta(1/2 + it)|^4 dt = (2\pi^2)^{-1} T\log^4 T + O(T\log^3 T).$$

This is a classical result of zeta-function theory, proved first by A.E. Ingham [1] by a difficult method, and the asymptotic formula (5.1) remained the best known mean value estimate of the zeta-function for a very long time, though for the somewhat easier problem of estimating $\int_0^\infty e^{-\delta t}|\zeta(1/2+it)|^4 dt$ $(\delta \to 0+)$ a sharp asymptotic formula has been obtained by F.V. Atkinson [2]. Recently D.R. Heath-Brown [3] improved substantially (5.1) by showing that

$$(5.2) \qquad \int_0^T |\zeta(1/2 + it)|^4 dt = T \sum_{k=0}^{4} c_k \log^{4-k} T + O(T^{7/8+\varepsilon}),$$

where $c_0 = (2\pi^2)^{-1}$, and the other c_k's are computable constants. As is to be expected the proof of (5.2) is long and difficult and will not be given here, but the proof of the classical result (5.1) may be given now in a relatively simple way by combining the reflection principle of Chapter 4 with the mean value theorem for Dirichlet polynomials. The mean value theorem for Dirichlet polynomials is a very useful result, which has two forms, discrete and integral. The integral variant of the theorem may be formulated as

THEOREM 5.2. Let $a_1,\dots,a_N$ be arbitrary complex numbers. Then

$$(5.3)\qquad \int_0^T \Big|\sum_{n\le N} a_n n^{it}\Big|^2 dt = T\sum_{n\le N}|a_n|^2 + O\Big(\sum_{n\le N} n|a_n|^2\Big),$$

and the above formula remains also valid if $N = \infty$, provided that the series on the right-hand side of (5.3) converge.

In §2 a proof of Theorem 5.2 and its discrete variant, Theorem 5.3, will be given, while in §3 a proof of (5.1) is presented.

§2. The mean value theorem for Dirichlet polynomials

We begin now the proof of Theorem 5.2. Squaring and integrating it is seen that the left-hand side of (5.3) equals

$$(5.4)\qquad T\sum_{n\le N}|a_n|^2 + \sum_{m\ne n\le N} a_m\bar{a}_n \frac{(m/n)^{iT}-1}{\log m-\log n},$$

so that (5.3) is a consequence of

$$(5.5)\qquad \sum_{m\ne n\le N}\frac{a_m\bar{a}_n}{\log m - \log n} \ll \sum_{n\le N} n|a_n|^2,$$

applied once directly and once with a_m replaced by $a_m m^{iT}$.

To obtain (5.5) we shall first prove

$$(5.6)\qquad \Big|\sum_{m\ne n}\frac{a_m\bar{a}_n}{m-n}\Big| \le \pi\sum |a_n|^2,$$

which is known in literature as Hilbert's inequality, and then deduce (5.5) from (5.6). In (5.6) the a's are arbitrary complex numbers, and m,n run over the same (possibly infinite) subset of the integers, subject only to the condition $m\ne n$. To see that (5.6) holds let

$$E = \sum_{m\ne n}\frac{a_m\bar{a}_n}{m-n}.$$

Then obviously $E = -\bar{E}$, which means that E is purely imaginary, hence E/i is real. Recalling that for integer k we have

$$(5.7)\qquad \int_0^1 e(kx)dx = \int_0^1 e^{2\pi ikx}dx = \begin{cases}1 & k=0\\ 0 & k\ne 0\end{cases},$$

it follows that

$$0 \leq \int_0^1\int_0^y \left|\sum a_n e(nx)\right|^2 dxdy = \int_0^1\int_0^y \left(\sum |a_n|^2 + \sum_{m\neq n} a_m \bar{a}_n e((m-n)x)\right)dxdy =$$

(5.8)

$$\frac{1}{2}\sum |a_n|^2 + \int_0^1 \sum_{m\neq n} a_m \bar{a}_n \frac{e((m-n)y) - 1}{2\pi i(m-n)}dy = \frac{1}{2}\sum |a_n|^2 - E/(2\pi i).$$

From (5.8) it follows that (5.6) holds if $E/i \geq 0$, and if $E/i < 0$ then the result follows if we repeat the above reasoning with $\left|\sum a_n e(-nx)\right|^2$ in place of $\left|\sum a_n e(nx)\right|^2$. The proof actually shows that we obtain

$$(5.9) \qquad \left|\sum_{m\neq n} \frac{a_m \bar{a}_n}{q_m - q_n}\right| \leq \pi \sum |a_n|^2$$

if $\{q_n\}_{n=1}^{\infty}$ is any sequence of integers such that $q_m \neq q_n$ if $m \neq n$. Moreover one has also

$$(5.10) \qquad \left|\sum_{m\neq n} \frac{a_m \bar{b}_n}{q_m - q_n}\right| \leq 3\pi\left(\sum |a_n|^2\right)^{1/2}\left(\sum |b_n|^2\right)^{1/2},$$

which follows from

$$(2\pi i)^{-1}\sum_{m\neq n} \frac{a_m \bar{b}_n}{q_m - q_n} = -\int_0^1\int_0^y \sum a_m e(q_m x)\sum \bar{b}_n e(-q_n x)\,dxdy + \frac{1}{2}\sum a_n \bar{b}_n$$

if one uses the Cauchy-Schwarz inequality, (5.8) and (5.9), since

$$\left|\int_0^1\int_0^y \sum a_m e(q_m x)\sum \bar{b}_n e(-q_n x)\,dxdy\right|^2 \leq$$

$$\int_0^1\int_0^y \left|\sum a_n e(q_n x)\right|^2 dxdy \cdot \int_0^1\int_0^y \left|\sum \bar{b}_n e(-q_n x)\right|^2 dxdy \leq \sum |a_n|^2 \sum |b_n|^2.$$

For simplicity of writing let now $L_n = \log n$ and

$$G = \sum_{m\neq n} \frac{a_m \bar{a}_n}{L_m - L_n},$$

so that G is purely imaginary and as in the proof of (5.6) it will be sufficient to assume that $G/i \geq 0$. From

$$0 \leq \int_0^1\int_0^1 \left|\sum a_n e(L_n x)\right|^2 dxdy = \frac{1}{2}\sum |a_n|^2 + \int_0^1 \sum_{m\neq n} a_m \bar{a}_n \frac{e((L_m - L_n)y) - 1}{2\pi i(L_m - L_n)} dy$$

we obtain then

$$(5.11) \qquad G/(2\pi i) \leq \frac{1}{2}\sum |a_n|^2 + \frac{1}{2\pi}\sum_{k,l\geq 1}\left|\sum_{(m,n)\in I_k \times I_l}\int_0^1 a_m \bar{a}_n \frac{e((L_m - L_n)y)}{L_m - L_n} dy\right|,$$

where the range of summation $1 \leq n \leq N$ in (5.5) has been divided into intervals $I_j = (N2^{-j}, N2^{1-j}]$, $j = 1,2,\ldots$, Since

$$\left|\int_0^1 e^{ixt}dx\right| = \left|\frac{e^{it} - 1}{t}\right| \leq 2|t|^{-1}, \quad (0 \neq t \in Re)$$

we have for $|k - l| \geq 2$ in (5.11)

$$\sum_{(m,n)\in I_k \times I_l}\int_0^1 a_m \bar{a}_n \frac{e((L_m - L_n)y)}{L_m - L_n} dy \quad \ll$$

$$\sum_{(m,n)\in I_k \times I_l} |a_m a_n| \max_{(m,n)\in I_k \times I_l} (L_m - L_n)^{-2} \quad \ll$$

$$(k - l)^2 \Big(\sum_{(m,n)\in I_k \times I_l} |a_m|^2\Big)^{1/2} \Big(\sum_{(m,n)\in I_k \times I_l} |a_n|^2\Big)^{1/2} \quad \ll$$

$$(k-l)^{-2} N2^{-(k+l)/2} \Big(\sum_{m\in I_k} |a_m|^2\Big)^{1/2} \Big(\sum_{n\in I_l} |a_n|^2\Big)^{1/2} \ll (k-l)^{-2}(S_k S_l)^{1/2},$$

where

$$S_j = \sum_{n\in I_j} n|a_n|^2.$$

Here we used the Cauchy-Schwarz inequality and

$$|L_m - L_n| \geq \log(N2^{-l}) - \log(N2^{1-k}) = \log 2 (k - l - 1) \geq \frac{1}{4}(k - l),$$

since $k - l \geq 2$, and the case $l - k \geq 2$ is analogous. Using again the Cauchy-Schwarz inequality we have

$$\sum_{|k-l|\geq 2}(k-l)^{-2}(S_kS_l)^{1/2} \leq \Big(\sum_{k\neq l}S_k(k-l)^{-2}\Big)^{1/2}\Big(\sum_{k\neq l}S_l(k-l)^{-2}\Big)^{1/2} \ll \sum_{n\leq N}n|a_n|^2,$$

since $\sum_{K,k\neq l}(k-l)^{-2}$ converges, so that the contribution of $\sum_{|k-l|\geq 2}$ to (5.11) is of the desired order of magnitude. The terms in (5.11) for which $|k-l|\leq 1$ may be written as

$$\int_0^1 \sum_{\substack{(m,n)\in I_k\times I_l\\ m\neq n}} a_m\bar{a}_n \frac{e((L_m-L_n)y)}{L_m-L_n}dy = M\int_0^1\Big(\sum_{\substack{(m,n)\in I_k\times I_l\\ m\neq n}}\frac{a'_m\bar{a}'_n}{ML_m-ML_n}\Big)dy,$$

where $a'_m = a_m e(L_m y)$, $M = N2^{6-k}$, and it will be sufficient to majorize the last sum. The reason for introducing M is that if $|k-l|\leq 1$, then for $m > n$ and $(m,n)\in I_k\times I_l$

$$[ML_m] - [ML_n] \geq M(L_m - L_n) - 2 = M\log\Big(1+\frac{m-n}{n}\Big) - 2 \geq$$

$$2^{6-k}N(m-n)/(3\cdot 2^{2-k}N) - 2 = \frac{16}{3}(m-n) - 2 \geq m-n,$$

since $m-n\geq 1$, $0\leq (m-n)/n\leq 3$ and $\log(1+x)\geq x/3$ for $0\leq x\leq 3$. Therefore we have for $|k-l|\leq 1$

$$(5.13)\quad M\sum_{\substack{(m,n)\in I_k\times I_l\\ m\neq n}}\frac{a'_m\bar{a}'_n}{ML_m-ML_n} = M\sum_{\substack{(m,n)\in I_k\times I_l\\ m\neq n}}\frac{a'_m\bar{a}'_n}{[ML_m]-[ML_n]} + O\Big(M\sum_{\substack{(m,n)\in I_k\times I_l\\ m\neq n}}\frac{|a_m a_n|}{(m-n)^2}\Big),$$

and the O-term above is easily seen to be $\ll (S_kS_l)^{1/2}$. The other term on the right-hand side of (5.13) is estimated by (5.10) with $q_m = [ML_m]$, and the resulting estimate is multiplied by M to yield also $\ll (S_kS_l)^{1/2}$. Using once again the Cauchy-Schwarz inequality and $|k-l|\leq 1$ we obtain (5.5) and consequently (5.3). If $N=\infty$ the reasoning is the same, only we define $I_j = (2^{j-1}, 2^j]$ this time.

We pass now to the discrete form of the mean value theorem for Dirichlet polynomials, which turns out to be more useful in certain applications than the integral form of the theorem. This may be formulated as

<u>THEOREM 5.3</u>. Let $1\leq t_1 < \ldots < t_R \leq T$ be real numbers such that $|t_r - t_s|\geq 1$ for $r\neq s\leq R$ and let $a_1,\ldots,a_N$ be arbitrary complex numbers. Then

$$(5.14)\qquad \sum_{r\le R}\Big|\sum_{n\le N} a_n n^{-it_r}\Big|^2 \ll \Big(T\sum_{n\le N}|a_n|^2 + \sum_{n\le N} n|a_n|^2\Big)\log N.$$

<u>Proof of Theorem 5.3.</u> If $0 \le x \le 1$ and $f(x)\in C^1[0,1]$,then an integration by parts shows that

$$f(x) = \int_0^1 f(t)dt + \int_0^x tf'(t)dt + \int_x^1 (t-1)f'(t)dt,$$

hence

$$(5.15)\qquad |f(1/2)| \le \int_0^1 (|f(x)| + \tfrac{1}{2}|f'(x)|)dx.$$

Taking $f(x) = F(x - 1/2 + t_r)$ we have from (5.15)

$$(5.16)\qquad |F(t_r)| \le \int_{t_r-1/2}^{t_r+1/2} |F(t)|dt + \frac{1}{2}\int_{t_r-1/2}^{t_r+1/2} |F'(t)|dt.$$

Now we use (5.16) with $F(t) = (\sum_{n\le N} a_n n^{-it})^2$. By the spacing condition imposed on the t_r's it is seen that the intervals $(t_r-1/2, t_r+1/2)$ $(r \le R)$ are disjoint, hence the left-hand side of (5.14) is

$$(5.17)\qquad \ll \int_0^T |F(t)|dt + \frac{1}{2}\int_0^T |F'(t)|dt.$$

The first integral in (5.17) is estimated directly by Theorem 5.2 and makes a contribution $\ll T\sum_{n\le N}|a_n|^2 + \sum_{n\le N}|a_n|^2 n$. For the other integral in (5.17) note that

$$F'(t) = -2\sum_{n\le N} a_n n^{-it}\sum_{n\le N} a_n \log n\cdot n^{-it},$$

hence by the Cauchy-Schwarz inequality and Theorem 5.2 we obtain

$$(5.18)\qquad \frac{1}{2}\int_0^T |F'(t)|dt \le \Big(\int_0^T \Big|\sum_{n\le N} a_n n^{it}\Big|^2 dt\Big)^{1/2}\Big(\int_0^T \Big|\sum_{n\le N} a_n \log n\cdot n^{it}\Big|^2 dt\Big)^{1/2} \ll$$

$$\Big(T\sum_{n\le N}|a_n|^2 + \sum_{n\le N} n|a_n|^2\Big)^{1/2}\Big(T\sum_{n\le N}|a_n|^2\log^2 n + \sum_{n\le N} n|a_n|^2\log^2 n\Big)^{1/2} \ll$$

$$\Big(T\sum_{n\le N}|a_n|^2 + \sum_{n\le N} n|a_n|^2\Big)\log N.$$

This completes the proof of Theorem 5.3, but it may be remarked that in the case $N = \infty$ the above proof gives

$$(5.19) \qquad \sum_{r\le R} \left|\sum_{n=1}^{\infty} a_n n^{-it_r}\right|^2 \ll T\sum_{n=1}^{\infty} |a_n|^2\log^2(n+1) + \sum_{n=1}^{\infty} n|a_n|^2\log^2(n+1),$$

provided that the series on the right-hand side converge.

Another remark is that from $|a_n| = |a_n n^{iT_0}|$, T_0 an arbitrary, fixed real number, it follows that Theorem 5.2 remains valid if $\int_0^T$ is replaced by $\int_{T_0}^{T_0+T}$, and similarly in Theorem 5.3 we may suppose that $T_0 + 1 \le t_1 < \ldots < t_R \le T_0 + T$.

§3. Proof of the fourth power moment estimate

As an application of Theorem 5.2 we shall present now a proof of Theorem 5.1 by using a variant of the reflection principle, which was discussed in Chapter 4. With $w = u + iv$, $s = \sigma + it$, u and v real, $0 < \sigma < \frac{3}{4}$, $T/2 \le t \le T$, we obtain from (1.7) on applying the residue theorem

$$\sum_{n=1}^{\infty} d(n)e^{-n/T}n^{-s} = (2\pi i)^{-1}\int_{u=2} \zeta^2(s + w)\Gamma(w)T^w dw =$$

$$\zeta^2(s) + O(T^{-c}) + (2\pi i)^{-1}\int_{u=-3/4} \chi^2(s + w)\zeta^2(1 - s - w)\Gamma(w)T^w dw =$$

$$(5.20) \quad \zeta^2(s) + O(T^{-c}) + (2\pi i)^{-1}\int_{u=-3/4} \chi^2(s + w)\sum_{n=1}^{\infty} d(n)n^{w+s-1}\Gamma(w)T^w dw =$$

$$\zeta^2(s) + O(T^{-c}) + (2\pi i)^{-1}\int_{u=-3/4} \chi^2(s + w)\sum_{n>T} d(n)n^{w+s-1}\Gamma(w)T^w -$$

$$-\chi^2(s)\sum_{n\le T} d(n)n^{s-1} + (2\pi i)^{-1}\int_{u=1/4} \chi^2(s + w)\sum_{n\le T} d(n)n^{w+s-1}\Gamma(w)T^w dw.$$

Here we used the functional equation (4.3) and Stirling's formula (1.32) to obtain the error term $O(T^{-c})$ (here $c > 0$ is arbitrary, but fixed) which majorizes the residue of $\zeta^2(s+w)\Gamma(w)T^w$ at the double pole $w = 1 - s$. Now we set $s = 1/2 + it$ and use again Stirling's formula together with

$$\zeta^2(1/2 + it)\chi^{-1}(1/2 + it) = |\zeta(1/2 + it)|^2$$

to deduce from (5.20)

$$(5.21) \qquad |\zeta(1/2 + it)|^2 = \sum_{k=1}^{6} J_k(s) + O(T^{-c})$$

for any fixed $c > 0$, where $J_k = J_k(s)$, and for $s = 1/2 + it$

$$J_2 = \bar{J}_1 = \chi(1/2 + it)\sum_{n\le T} d(n)n^{-1/2+it},$$

$$J_3 = \chi^{-1}(1/2 + it)\sum_{n>T} d(n)e^{-n/T}n^{-1/2-it},$$

$$J_4 = \chi^{-1}(1/2 + it)\sum_{n\le T} d(n)(e^{-n/T} - 1)n^{-1/2-it},$$

$$J_5 = -(2\pi i)^{-1}\chi^{-1}(1/2+it) \int\limits_{u=-3/4,\ |v|\le \log^2 T} \chi^2(1/2+it+w)\sum_{n>T} d(n)n^{w-1/2+it}\Gamma(w)T^w dw,$$

$$J_6 = -(2\pi i)^{-1}\chi^{-1}(1/2+it) \int\limits_{u=1/4,\ |v|\le \log^2 T} \chi^2(1/2+it+w)\sum_{n\le T} d(n)n^{w-1/2+it}\Gamma(w)T^w dw.$$

Theorem 5.1 will follow then from

$$(5.22) \qquad \int_{T/2}^{T} |\zeta(1/2 + it)|^4 dt = (4\pi^2)^{-1}T\log^4 T + O(T\log^3 T),$$

when one replaces T by $T/2, T/2^2$, ... etc. and adds all the results. Observe that trivially $J_k \ll T^{1/2}\log T$ for each k. Therefore squaring and integrating (5.21) we have

$$(5.23) \qquad \int_{T/2}^{T} |\zeta(1/2 + it)|^4 dt = 2\int_{T/2}^{T} |J_1|^2 dt + \int_{T/2}^{T} (J_1^2 + J_2^2)dt + $$
$$+ O\Big(\sum_{k=3}^{6} \int_{T/2}^{T} |J_k|^2 dt\Big) + O\Big(\sum_{k=3}^{6} \Big|\int_{T/2}^{T} (J_1 + J_2)J_k dt\Big|\Big) + O(1),$$

and the main contribution in (5.22) will come from the first integral on the right-hand side of (5.23). To see this note that from the Dirichlet series representations

$$\sum_{n=1}^{\infty} d^2(n)n^{-s} = \zeta^4(s)/\zeta(2s), \quad \sum_{n=1}^{\infty} \mu(n)n^{-2s} = 1/\zeta(2s)$$

which are valid for $\mathrm{Re}\, s > 1$ and $\mathrm{Re}\, s > 1/2$ respectively, one obtains by an easy convolution argument

$$(5.24) \qquad \sum_{n\le x} d^2(n) = \pi^{-2}x\log^3 x + O(x\log^2 x),$$

and hence by partial summation

$$(5.25)\qquad \sum_{n\le x} d^2(n)n^a = \begin{cases} c(a)x^{1+a}\log^3 x + O(x^{1+a}\log^2 x) & a \ne -1, \\ (4\pi^2)^{-1}\log^4 x + O(\log^3 x) & a = -1. \end{cases}$$

Now we shall apply Theorem 5.2 and (5.25), obtaining first

$$(5.26)\qquad 2\int_{T/2}^{T} |J_1|^2 dt = T\sum_{n\le T} d^2(n)n^{-1} + O(\sum_{n\le T} d^2(n)) = (4\pi^2)^{-1}T\log^4 T + O(T\log^3 T),$$

since $|\chi(1/2\pm it)| = 1$. Therefore (5.26) does contribute the main term in (5.22), and in fact the main idea of the proof is to apply Theorem 5.2 to the remaining integrals in (5.23) using (5.25) with $a \ne -1$. Thus we have

$$\int_{T/2}^{T} |J_3|^2 dt \ll T\sum_{n>T} d^2(n)e^{-2n/T}n^{-1} + \sum_{n>T} d^2(n)e^{-2n/T} \ll T^{\varepsilon},$$

$$\int_{T/2}^{T} |J_4|^2 dt \ll T\sum_{n\le T} d^2(n)(e^{-n/T}-1)^2 n^{-1} + \sum_{n\le T} d^2(n)(e^{-n/T}-1)^2 \ll$$

$$T^{-1}\sum_{n\le T} d^2(n)n + T^{-2}\sum_{n\le T} d^2(n)n^2 \ll T\log^3 T,$$

where we used $e^{-x} - 1 \le x$ for $x \ge 0$,

$$\int_{T/2}^{T} |J_5|^2 dt \ll T^{5/2}\sum_{n>T} d^2(n)n^{-5/2} + T^{3/2}\sum_{n>T} d^2(n)n^{-3/2} \ll T\log^3 T,$$

$$\int_{T/2}^{T} |J_6|^2 dt \ll T^{1/2}\sum_{n\le T} d^2(n)n^{-1/2} + T^{-1/2}\sum_{n\le T} d^2(n)n^{1/2} \ll T\log^3 T.$$

Next we write

$$(5.27)\qquad i\int_{T/2}^{T} J_1^2 dt = \int_{1/2+iT/2}^{1/2+iT} J_1^2(s)ds$$

and consider the last integral as an integral of the complex variable s. To avoid (5.25) with $a = -1$ we replace by the residue theorem the segment of integration in (5.27) by segments joining the points $1/2 + iT/2, 1/4 + iT/2, 1/4 + iT, 1/2 + iT$. Using $\chi(s) \asymp T^{1/2-\sigma}$ it is seen that the integrals over horizontal segments are $\ll T\log^2 T$, while

$$\int_{1/4+iT/2}^{1/4+iT} J_1^2(s)ds \ll T^{-1/2}\int_{T/2}^{T} \Big|\sum_{n\le T} d(n)n^{-1/4+it}\Big|^2 dt \ll T\log^3 T$$

on using Theorem 5.2 and (5.25). The same procedure may be applied to the integral of J_2^2 to yield

$$(5.28)\qquad \int_{T/2}^{T} (J_1^2 + J_2^2)dt \ll T\log^3 T.$$

The remaining integrals in (5.23) are written as

$$i \int_{1/2+iT/2}^{1/2+iT} (J_1(s) + J_2(s))J_k(s)ds, \quad (k = 3,4,5,6)$$

and are treated similarly. In integrals with $J_1(s)$ the segment $[1/2+iT/2, 1/2+iT]$ is being replaced by the segment $[3/8+iT/2, 3/8+iT]$ with an error $\ll T\log^3 T$, while in integrals containing $J_2(s)$ it is replaced by the segment $[5/8+iT/2, 5/8+iT]$ with an error $\ll T\log^3 T$ also. Applying the Cauchy-Schwarz inequality, Theorem 5.2 and collecting all the estimates we obtain then as asserted

$$(5.29) \qquad \int_{T/2}^{T} |\zeta(1/2 + it)|^4 dt = (4\pi^2)^{-1} T\log^4 T + O(T\log^3 T),$$

so that (5.1) follows from (5.29) on replacing T by $T/2, T/2^2, \ldots$ etc. and adding all the results.

N O T E S

Various mean value estimates for $\int_1^T |\zeta(\sigma + it)|^k dt$ are discussed in Chapter 7 of Titchmarsh [8], but (5.1) is not proved there, only the weaker formula

$$(5.30) \qquad \int_0^T |\zeta(1/2 + it)|^4 dt = (1 + o(1))(2\pi^2)^{-1} T\log^4 T.$$

This follows from investigation of the integrals

$$I(T) = \int_0^T |\zeta(\sigma + it)|^{2k} dt, \quad J(\delta) = \int_0^\infty |\zeta(\sigma + it)|^{2k} e^{-\delta t} dt,$$

where $k \geq 1$ is a fixed integer, $T \to \infty$ and $\delta \to 0+$, $1/2 \leq \sigma < 1$ is fixed. A simple Tauberian argument shows that, for $C > 0$, $I(T) \sim CT\log^D T$ is equivalent with $J(\delta) \sim C\delta^{-1}(\log\delta^{-1})^D$ $(D > 0)$, and Titchmarsh then deduces (5.30) from

$$\int_0^\infty |\zeta(1/2 + it)|^4 e^{-\delta t} dt = (1 + o(1))(2\pi^2)^{-1}\delta^{-1}(\log\delta^{-1})^4, \quad \delta \to 0+.$$

As mentioned in §1, a sharper result has been obtained by F.V. Atkinson [2], who proved

$$(5.31) \quad \int_0^\infty |\zeta(1/2 + it)|^4 e^{-\delta t} dt = \sum_{i=0}^{4} A_i \delta^{-1}(\log\delta^{-1})^{4-i} + O((\delta^{-1})^{13/14+\epsilon}),$$

where $A_0 = (2\pi^2)^{-1}$ and the other constants are computable. A method is also indicated in Atkinson's paper by which the exponent 13/14 may be reduced to 8/9. However (5.31) does not seem to imply (5.1), but only the weaker (5.30). Similarly it may be mentioned that one has (Theorem 7.15 (A) of Titchmarsh [8]), as $\delta \to 0+$,

$$\int_0^\infty |\zeta(1/2 + it)|^2 e^{-2\delta t}dt = \frac{\gamma - \log(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^{N} c_n\delta^n + O(\delta^{N+1}),$$

for any fixed integer $N \geq 1$, but this sharp result does not seem to imply anything like Atkinson's formula for $\int_0^T |\zeta(1/2 + it)|^2 dt$, which will be extensively discussed in Chapter 11.

Heath-Brown's proof of (5.2) in [3] is based on several ideas. The first is the use of an approximate functional equation which may be written as

$$(5.32) \quad |\zeta(1/2 + it)|^{2k} = \sum_{mn \leq cT^k} d_k(m)d_k(n)(mn)^{-1/2}(m/n)^{it}K(mn,t) + O(T^{-2}).$$

Here $k \geq 1$ is a fixed integer, $c \geq 1$ is a constant depending on k, $T \leq t \leq 2T$,

$$(5.33) \quad K(x,t) = (\pi i)^{-1} \int_{1-i\infty}^{1+i\infty} t^{kz}(2\pi)^{-kz}x^{-z}e^{z^2T^{-1}}\left(1 + \sum_{u=1}^{U} \sum_{v} \alpha(u,v)z^u t^{-2v}\right)z^{-1}dz,$$

where $\alpha(u,v)$ is a constant and U an integer depending on k, while $\sum_v$ denotes summation for $\max(1,u/3) \leq v \leq U$.

This result may be compared with (4.25), the approximate functional equation for $\zeta^k(s)$ of Chapter 4. The main terms in (5.32) are much more complicated than the main terms in (4.25), but in contrast with (4.26) and (4.27) the error term $O(T^{-2})$ given here by Heath-Brown is very sharp. This enabled him to integrate (5.32) with k = 2 termwise, but there were difficulties which arose from the dependence of K(x,t) on t. A further feature of the proof of (5.2) is the use of an exponential averaging technique, which permits one to evaluate $\int_{2T}^{4T} |\zeta(1/2+it)|^4 dt$ using the weighted integral $\int_T^{5T} w(t)|\zeta(1/2+it)|^4 dt$, where the function w(t) is precisely defined in §3 of Heath-Brown [3]. The proof in its last stage requires an asymptotic formula for the divisor sum

$$D(x,r) = \sum_{n\le x} d(n)d(n+r) = m(x,r) + E(x,r),$$

where r may be increasing with x. Here m(x,r) is the main term of the form

$$m(x,r) = \sum_{j=0}^{2} c_j(r)x\log^j x,$$

and the exponent 7/8 in (5.2) is mostly limited by the mean value estimate

$$\int_X^{2X} E^2(x,r)dx \ll X^{5/2+\varepsilon},$$

which holds uniformly in r for $r \le X^{3/4}$.

The proof of Theorem 5.2 is based on K. Ramachandra [5], and the crucial estimate (5.5) is a special case of a more general inequality due to H.L. Montgomery and R.C. Vaughan [1]: suppose that $R \ge 2$ and $\lambda_1, \lambda_2, \ldots, \lambda_R$ are distinct real numbers such that $0 < \delta_n = \min_{m\ne n}|\lambda_n - \lambda_m|$. If $a_1, a_2, \ldots, a_R$ are arbitrary complex numbers, then

$$\left|\sum_{m\ne n} a_m\bar{a}_n(\lambda_m - \lambda_n)^{-1}\right| \le \tfrac{3}{2}\pi \sum_n |a_n|^2\delta_n^{-1}. \tag{5.34}$$

This inequality is closely connected with large sieve type inequalities for which the reader may consult the expository paper of H.L. Montgomery [5].

In presenting the proof of Theorem 5.1 we have followed the work of K. Ramachandra [3]. Ramachandra's method does not seem to extend to give anything sharper than (5.1), yet it is incomparably simpler than the method used by A.E. Ingham [1] in proving (5.1).

For other mean value theorems for Dirichlet polynomials the reader may consult H.L. Montgomery [2], Chapters 6 and 7.

In estimating the first sum on the right-hand side of (5.13) by (5.10) we take $a_m = a'_m$ for $m \in I_k$ and zero otherwise, $b_n = a'_n$ for $n \in I_1$ and zero otherwise. Then the sum in question is

$$\ll M\Big(\sum_{m\in I_k} |a_m|^2\Big)^{1/2}\Big(\sum_{n\in I_1} |a_n|^2\Big)^{1/2} \ll$$

$$\ll \Big(\sum_{m\in I_k} m|a_m|^2\Big)^{1/2}\Big(\sum_{n\in I_l} n|a_n|^2\Big)^{1/2} = (S_k S_l)^{1/2},$$

as asserted.

Concerning the convolution argument that leads to (5.24), observe that from the Dirichlet series representation (or directly) one has

$$d^2(n) = \sum_{kl^2=n} d_4(k)\mu(l),$$

hence

$$\sum_{n\le x} d^2(n) = \sum_{kl^2\le x} d_4(k)\mu(l) = \sum_{l\le x^{1/2}} \mu(l) \sum_{k\le xl^{-2}} d_4(k) =$$

$$\sum_{l\le x^{1/2}} \mu(l)\left(\tfrac{1}{6}xl^{-2}\log^3(xl^{-2}) + O(xl^{-2}\log^2 x)\right) =$$

$$\pi^{-2}x\log^3 x + O(x\log^2 x).$$

Here we used

$$\sum_{n=1}^{\infty}\mu(n)n^{-2} = 1/\zeta(2) = 6\pi^{-2},$$

and the weak asymptotic formula

$$\sum_{n\le x} d_4(n) = \tfrac{1}{6}x\log^3 x + O(x\log^2 x).$$

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TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 6

MEAN VALUE ESTIMATES OVER SHORT INTERVALS

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MEAN VALUE ESTIMATES OVER SHORT INTERVALS

§1. Introduction

The proof of the fourth power moment estimate (5.1) depended on an approximate functional equation for $\zeta^2(s)$ (see (4.66)), and even Heath-Brown's proof [3] of the much stronger result (5.2) also depended on another approximate functional equation, namely (5.32) and (5.33). Thus the natural line of approach in estimating $\int_0^T |\zeta(\sigma+it)|^k dt$ $(\sigma \geq 1/2, k > 4)$ would be to use an approximate functional equation for $\zeta^k(s)$ and then to integrate it termwise. However to this day no satisfactory result based on this idea has been obtained, and in this chapter we focus our attention on integrals of the type $\int_{T-G}^{T+G} |\zeta(\sigma+it)|^2 dt$, where $1/2 \leq \sigma < 1$ is fixed. The interval of integration is "short" in the sense that we shall always suppose $G = o(T)$ as $T \to \infty$, and the purpose of estimating this type of integrals will be seen in Chapter 7, where they will be used for estimates of $\int_0^T |\zeta(\sigma+it)|^k dt$ $(k > 4)$. This idea was first used by D.R. Heath-Brown [1] in his proof of the twelfth power moment estimate (7.15). His proof of the crucial estimate (this is essentially our Theorem 6.2) depended on the deep formula of F.V. Atkinson [3] for $\int_0^T |\zeta(1/2+it)|^2 dt$ which will be discussed in Chapter 11. The proof of Theorem 6.2 that will be given here is new and dispenses completely with Atkinson's formula. Besides the approximate functional equation for $\zeta^2(s)$ it uses Voronoï's summation formula and Atkinson's saddle point result (Theorem 2.2). This line of attack on power moments for the zeta-function is motivated by M. Jutila's paper [6], and it shows that power moment estimates for the zeta-function for $k > 4$ may be made independent of Atkinson's formula [3].

§2. An auxiliary estimate

To facilitate subsequent estimates we shall start with a technical lemma which is a straightforward generalization of a lemma due to Heath-Brown [1]. Its significance lies in the fact that it estimates the moduli of the zeta-function by an integral of nearly the same function. This is

Lemma 6.1. Let $k \geq 1$ be a fixed integer and $T/2 \leq t \leq 2T$. Then for $1/2 \leq \sigma < 1$ fixed we have uniformly in t

$$(6.1) \qquad |\zeta(\sigma + it)|^k \ll 1 + \log T \int_{-\log^2 T}^{\log^2 T} |\zeta(\sigma - 1/\log T + it + iv)|^k e^{-|v|} dv,$$

and

$$(6.2) \qquad |\zeta(1/2 + it)|^k \ll \log T\Big(1 + \int_{-\log^2 T}^{\log^2 T} |\zeta(1/2 + it + iv)|^k e^{-|v|} dv\Big).$$

Proof of Lemma 6.1. Let first $1/2 \leq \sigma < 1$ be fixed, $c = 1/\log T$, $s' = \sigma + c + it$. From (1.7) we obtain by termwise integration

$$(6.3) \qquad (2\pi i)^{-1} \int_{1-i\infty}^{1+i\infty} \zeta^k(s' + w)\Gamma(w)dw = \sum_{n=1}^{\infty} d_k(n)e^{-n}n^{-s'} \ll 1.$$

Moving the line of integration in (6.3) to $\mathrm{Re}\, w = -c$ we encounter poles at $w = 1 - s$ (of order k) and $w = 0$ with residues $O(1)$ (in view of (1.32)) and $\zeta^k(s')$ respectively. Since $s = 0$ is a simple pole of $\Gamma(s)$ then also in view of (1.32) we have then for any real v

$$(6.4) \qquad \Gamma(\pm c \pm iv) \ll e^{-|v|}(c + |v|)^{-1},$$

so that (6.3) yields for $T/3 \leq t \leq 3T$

$$(6.5) \qquad \zeta^k(s') \ll 1 + \int_{-\infty}^{\infty} |\zeta(\sigma + it + iv)|^k e^{-|v|}(c + |v|)^{-1} dv.$$

To obtain (6.1) from (6.5) we only have to note that $c^{-1} = \log T$ and that for any fixed $A > 0$

$$\int_{\pm\log^2 T}^{\pm\infty} |\zeta(\sigma + it + iv)|^k e^{-|v|}(c + |v|)^{-1} dv \ll \int_{\log^2 T}^{\infty} e^{-v/2} dv \ll T^{-A},$$

and finally we have to replace σ by $\sigma - c$ in (6.5).

Now we suppose that $\sigma = 1/2$ and note that by the functional equation

$$|\zeta(1/2 - c + it)| \ll |\zeta(1/2 + c + it)|T^c \ll |\zeta(1/2 + c + it)|,$$

so that (6.5) remains true if $\sigma = 1/2$, $s' = 1/2 - c + it$. On the other hand by the residue theorem we have for $s = 1/2 + it$

$$\zeta^k(s) = (2\pi i)^{-1}\int_D \zeta^k(s+z)\Gamma(z)\,dz, \tag{6.6}$$

where D is the rectangle with vertices $\pm c \pm i\log^2 T$. Using Stirling's formula (1.32) it is seen that the integrals over the horizontal sides if D are $o(1)$, and using (6.5) with $\delta = 1/2$, $s' = 1/2 \pm c \pm i(t+u)$, $|u| \le \log^2 T$, we obtain

$$\zeta^k(s) \ll 1 + \int_{-\log^2 T}^{\log^2 T} e^{-|u|}\Big(1 + \int_{-\infty}^{\infty} |\zeta(1/2+it+iu+iv)|^k (c+|v|)^{-1} e^{-|v|}dv\Big)(c+|u|)^{-1}du. \tag{6.7}$$

To estimate the above expression first note that trivially

$$\int_{-\log^2 T}^{\log^2 T} e^{-|u|}(c+|u|)^{-1}du \ll c^{-1} = \log T,$$

and in the remaining integral we make the substitution $v = x - u$ and invert the order of integration. This gives

$$\zeta^k(1/2+it) \ll \log T + \int_{-\infty}^{\infty} |\zeta(1/2+it+ix)|^k \Big(\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|u|)^{-1}(c+|x-u|)^{-1}du\Big)dx, \tag{6.8}$$

and the proof of (6.2) will be finished if we can show

$$\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|u|)^{-1}(c+|x-u|)^{-1}du \ll c^{-1}e^{-|x|}. \tag{6.9}$$

This is obvious when $x = 0$, and as the cases $x > 0$ and $x < 0$ are treated analogously, we shall consider $x > 0$ only. Write

$$\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|u|)^{-1}(c+|x-u|)^{-1}du = \int_{-\infty}^{0} + \int_{0}^{x} + \int_{x}^{\infty} = I_1 + I_2 + I_3,$$

say. Then

$$I_1 = \int_0^\infty e^{-x}(c+v)^{-1}(c+x+v)^{-1}dv \ll e^{-x}\Big(\int_0^c c^{-2}dv + \int_c^\infty v^{-2}dv\Big) \ll e^{-x}c^{-1},$$

$$I_2 = \int_0^x e^{-x}(c+u)^{-1}(c+x-u)^{-1}du \ll e^{-x}\int_0^{x/2}(c+u)^{-2}du \ll e^{-x}c^{-1},$$

since $c + u = c + x - u$ for $u = x/2$, and finally

$$I_3 = \int_x^{\infty} e^{-2u+x}(c+u)^{-1}(c+u-x)^{-1}du \le \tfrac{1}{2}e^{-x}\int_x^{\infty}\{(c+u)^{-2} + (c+u-x)^{-2}\}du \ll c^{-1}e^{-x}.$$

§3. The mean square when σ is in the critical strip

We consider now the integral

$$\int_{T-G}^{T+G} |\zeta(\sigma + it)|^2 dt, \quad G = o(T), \quad 1/2 < \sigma < 1,$$

where σ is fixed, leaving aside the most important case $\sigma = 1/2$ for the next section. We need the following lemma, whose proof is typical of several proofs in the sequel and uses the exponential integral (1.34) to "shorten" exponential sums under consideration.

Lemma 6.2. For $N < N' \le 2N \ll T^A$, $A > 0$ fixed, $\log T < G \le T$, we have uniformly in G

(6.10)
$$\int_{T-G}^{T+G} \Big|\sum_{N<n\le N'} n^{-it}\Big|^2 dt \ll$$

$$NG\log T + G \sum_{r \le NG^{-1}\log T} \max_{N<n\le N''\le N'-r} \Big|\sum_{N<n\le N''} \exp(iT\log(1 + r/n))\Big|.$$

Proof of Lemma 6.2.

(6.11)
$$\int_{T-G}^{T+G} \Big|\sum_{N<n\le N'} n^{-it}\Big|^2 dt \le e\int_{-G\log T}^{G\log T} \Big|\sum_{N<n\le N'} n^{-it-iT}\Big|^2 \exp(-t^2G^{-2})dt \ll$$

$$NG\log T + \Big|\sum_{N<m\ne n\le N'} (m/n)^{-iT}\int_{-\infty}^{\infty} \exp(-it(\log m/n) - t^2G^{-2})dt\Big| + o(1),$$

since

$$\int_{\pm G\log T}^{\pm\infty} \exp(-t^2G^{-2})dt \ll \exp(-\tfrac{1}{2}\log^2 T) \ll T^{-C}$$

for any fixed $C > 0$, and N is bounded by a fixed power of T. Because of symmetry we may suppose that $m > n$ in the last sum in (6.11) and use (1.34) to obtain

(6.12)
$$\sum_{N<n<m\le N'} (m/n)^{-iT}\int_{-\infty}^{\infty} \exp(-it(\log m/n) - t^2G^{-2})dt =$$

$$\pi^{1/2}G \sum_{N<n<m\le N'} \exp(-iT(\log m/n))\exp(-\tfrac{1}{4}G^2\log^2 m/n) =$$

$$\pi^{1/2}G \sum_{r\le NG^{-1}\log T}\ \sum_{N<n\le N'-r} \exp(-iT\log(1+r/n))\exp(-\tfrac{1}{4}G^2\log^2(1+r/n)) + o(1),$$

since writing $m = n + r$ we see that for $r > NG^{-1}\log T$

$$\exp(-\tfrac{1}{4}G^2\log^2 m/n) \le \exp(-G^2r^2/16N^2) \le \exp(-\tfrac{1}{16}\log^2 T) \le T^{-C}$$

for T sufficiently large and any fixed $C > 0$, because $\log(1+x) \ge x/2$ for $0 \le x \le 1$. The lemma now follows easily from (6.11),(6.12) and partial summation; if the sums on the right-hand side of (6.10) are empty they shall be of course counted as zero.

We proceed now with the main result of this section, whose proof will follow easily from Lemma 6.1 with the use of the approximate functional equation.

<u>THEOREM 6.1</u>. Let (p,q) be an exponent pair and $1/2 < \sigma < 1$ fixed. Then for $T^{(p+q+1-2\sigma)/2(p+1)}(\log T)^{(2+p)/(p+1)} \le G \le T$, $1 + q - p \ge 2\sigma$, we have uniformly in G

$$(6.13)\qquad \int_{T-G}^{T+G} |\zeta(\sigma + it)|^2 dt \ll G.$$

<u>Proof of Theorem 6.1</u>. From the approximate functional equation (4.10) we have with $x = y = (t/2\pi)^{1/2}$

$$(6.14)\qquad \zeta(\sigma+it) \ll 1 + \Big|\sum_{n\le(T/2\pi)^{1/2}} n^{-\sigma-it}\Big| + T^{1/2-\sigma}\Big|\sum_{n\le(T/2\pi)^{1/2}} n^{\sigma-1-it}\Big|,$$

where the error made by replacing $(t/2\pi)^{1/2}$ by $(T/2\pi)^{1/2}$ in the range of summation is clearly $O(1)$ if $T - G \le t \le T + G$ and $G \le T^{(1+\sigma)/2}$. For the less interesting range $T^{(1+\sigma)/2} < G \le T$ the theorem follows from the approximate functional equation (4.66) (where the lengths of the sums involved do not depend on t) and the mean value theorem (5.2) for Dirichlet polynomials. The intervals of summation in (6.14) are split into $O(\log T)$ subintervals of the form $(N,2N]$, $N = [(T/2\pi)^{1/2}]2^{-j}$, $j = 1,2,\ldots$, and thus by partial summation (equation (1.17))

$$(6.15)\qquad \int_{T-G}^{T+G} |\zeta(\sigma+it)|^2 dt \ll G + \sum_N \max_{N<N'\le 2N} N^{-2\sigma} \int_{T-G}^{T+G} \Big|\sum_{N<n\le N'} n^{-it}\Big|^2 dt,$$

since $\sigma > 1/2$, $N \ll T^{1/2}$. For $N \le T^{\varepsilon}$ we use again the mean value theorem for Dirichlet polynomials, and for $N > T^{\varepsilon}$ we use Lemma 6.2, which leads to the estimation of the exponential sum

$$(6.16)\qquad S = \sum_{N<n\le N'} \exp(iT\log(1 + r/n)) = \sum_{N<n\le N'} \exp(if(n)),$$

where for r,T fixed $f(x) = T\log(1 + r/x)$, $N \le x \le 2N$. The condition $N \ll T^{1/2}$ ensures that $f'(x) \gg 1$ for $N \le x \le 2N$, and in the same range we have also

$$f^{(k)}(x) \asymp TrN^{-k-1},\ k = 1,2,\ldots,$$

so that we may use the theory of exponent pairs, as presented in §3 of Chapter 2, to estimate S. We obtain

$$(6.17)\qquad S \ll \max_{N\le x\le 2N} |f'(x)|^p N^q \ll T^p r^p N^{q-2p}$$

for any exponent pair (p,q). Therefore combining (6.14)-(6.17) and Lemma 6.2, we obtain in view of $T^{\varepsilon} < N \ll T^{1/2}$, $1 + q - p \ge 2\sigma$,

$$(6.18)\qquad \int_{T-G}^{T+G} |\zeta(\sigma+ it)|^2 dt \ll G + GT^{\varepsilon(1-2\sigma)}\log T + G\sum_N \sum_{r\le NG^{-1}\log T} T^p r^p N^{q-2p-2\sigma} \ll$$

$$\ll G + G\sum_N T^p (NG^{-1}\log T)^{1+p} N^{q-2p-2\sigma} \ll$$

$$G + \log T\cdot\max_N G^{-p} T^p N^{1+q-p-2\sigma} \log^{1+p} T \ll G,$$

for $G \ge T^{(p+q+1-2\sigma)/2(p+1)} (\log T)^{(2+p)/(1+p)}$, proving Theorem 6.1.

From the approximate functional equation one obtains the well-known relation (see also Titchmarsh [8], Chapter 7)

$$\int_0^T |\zeta(\sigma + it)|^2 dt = (\zeta(2\sigma) + o(1))T,\ (1/2 < \sigma < 1)$$

which shows that the bound in (6.13) is of the expected order of magnitude, and a sharper asymptotic formula than the one above is given by (7.99).

Following the proof of Theorem 6.1 in case when $\sigma = 1/2$ we arrive at

$$(6.19)\qquad \int_{T-G}^{T+G} |\zeta(1/2+it)|^2 dt \ll G\log T,\quad T^{(p+q)/2(p+1)}(\log T)^{(p+2)/(p+1)} \le G \le T,$$

but in the next section we shall obtain an estimate which improves (6.19).

§4. The mean square when $\sigma = 1/2$

The theorem which will be stated and proved in this section is one of the fundamental results of this text, since it serves as a basis for the derivation of higher power moments of the zeta-function and provides a technically simple way of estimating $\zeta(1/2 + iT)$. The result is due to D.R. Heath-Brown [1], who used the averaging integral (1.34) and the deep formula of F.V. Atkinson [3]. As mentioned in §1, the proof that will be presented here is new and self-contained in the sense that it does not in any way depend on Atkinson's result, but is based on the approximate functional equation for $\zeta^2(s)$ and Voronoï's formula, as suggested by M. Jutila's work [6].

THEOREM 6.2. For $T^{\varepsilon} \le G \le T^{1/2-\varepsilon}$ uniformly in G

$$(6.20)\qquad \int_{T-G}^{T+G} |\zeta(1/2+it)|^2 dt \ll G\log T + G\sum_K (TK)^{-1/4}\Big(|S(K)| + K^{-1}\int_0^K |S(x)|\,dx\Big)e^{-G^2K/T},$$

where

$$(6.21)\qquad S(x) = S(x,K,T) = \sum_{K<n\le K+x} (-1)^n d(n)\exp(if(T,n)),$$

$$(6.22)\qquad f(T,n) = 2T\operatorname{ar\,sinh}((\pi n/2T)^{1/2}) + (\pi^2 n^2 + 2\pi nT)^{1/2},$$

and summation is over $K = 2^k$ such that $T^{1/3} \le K \le N$, where for $\delta > 0$ fixed

$$(6.23)\qquad N = B^2/(T/2\pi - B),\ B = T(2\pi G)^{-1}\log^{(1+\delta)/2} T.$$

Proof of Theorem 6.2. From (6.23) we have $K \ll TG^{-2}\log^{1+\delta} T$, and the interesting range for K is $K \ge T^{1/3}$, since the trivial bound $S(x) \ll K\log T$ gives

$$\sum_j \sum_{2^{-j}N=K\le T^{1/3}} (TK)^{-1/4}\Big(|S(K)| + K^{-1}\int_0^K |S(x)|\,dx\Big)e^{-G^2K/T} \ll$$

$$\sum_j \sum_{2^{-j}N=K\le T^{1/3}} \log T\cdot T^{-1/4}K^{3/4}e^{-G^2K/T} \ll \log T,$$

and thus it is seen that the relevant range for G is $G \le T^{1/3}\log T$, since $T^{1/3} \le K \ll TG^{-2}\log^{1+\delta}T$. Another remark is that in view of the exponential factor $\exp(-G^2K/T)$ the proof that will be given actually shows that uniformly for $T/2 \le \sigma \le T$ and $T^{\varepsilon} \le G \le T^{1/2-\varepsilon}$ we obtain

$$(6.24)\qquad \int_{\sigma-G}^{\sigma+G} |\zeta(1/2+it)|^2 dt \ll G\log T + G\sum_K (TK)^{-1/4}\left(|S(K)| + K^{-1}\int_0^K |S(x)|\,dx\right)e^{-G^2K/(2T)},$$

where $S(x) = S(x,K,\sigma)$ is given by (6.21), $T^{1/3} \le K = 2^k \le N$, and N is given by (6.23). This form of the mean value estimate will be particularly useful for higher power moment estimates in Chapter 7.

Since $\operatorname{ar\,sinh} x = (1 + o(1))x$ as $x \to 0$, it is seen that (6.20) will follow readily by partial summation from

$$(6.25)\qquad \int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt \ll G\log T +$$

$$+ G\left|\sum_{n\le N}(-1)^n d(n)n^{-1/2}(1/4+T/2\pi n)^{-1/4}\exp(-(G\operatorname{arsinh}((\pi n/2T)^{1/2}))^2)\exp(if(T,n))\right|,$$

and so we set out to prove (6.25). To facilitate the notation we introduce the abbreviations

$$(6.26)\qquad T' = T/(2\pi),\quad L = \log T,\quad I = \int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt.$$

The first step is similar to the one made in the proof of Theorem 6.1, and consists in majorizing I by a "short" exponential sum (of length $\ll TL/G$). We start from the approximate functional equation (4.13), which gives

$$(6.27)\qquad I \ll \int_{-GL}^{GL} |\zeta(1/2+it+iT)|^2 \exp(-t^2G^{-2})dt \ll GL + \int_{-GL}^{GL}(S_1 + \bar{S}_1)\exp(-t^2G^{-2})dt,$$

where

$$(6.28)\qquad S_1 = \chi(1/2+it+iT)\sum_{n\le(T+t)/2\pi} d(n)n^{-1/2+it+iT} = O(L) + \sum_{n\le T'} d(n)n^{-1/2+iq(T+t)},$$

$$(6.29)\qquad q(x) = x\log(2\pi n/x) + x + \pi/4,$$

if we utilize the asymptotic formula (4.4). For $-GL \le t \le GL$ we have by Taylor's formula

$$(6.30) \qquad q(T+t) = q(T) + t\log(2\pi n/T) - t^2/(2T) + O(G^3L^3T^{-2}),$$

so that from (6.27)-(6.30) we obtain

$$(6.31) \qquad I \ll GL + I_1 + \bar{I}_1,$$

where

$$(6.32) \qquad I_1 = \int_{-GL}^{GL} \sum_{n\le T'} d(n)n^{-1/2+iq(T)}\exp(it(\log n/T')-it^2/(2T)-t^2G^{-2})dt +$$

$$+ O(G^4L^5T^{-3/2}) = \sum_{n\le T'} d(n)n^{-1/2+iq(T)}\int_{-\infty}^{\infty}\exp(it(\log n/T')-it^2/(2T)-t^2G^{-2})dt + O(GL),$$

since $G \le T^{1/2-\varepsilon}$ and $\int_{\pm GL}^{\pm\infty} \ll T^{-c}$ for any fixed $c > 0$. The last integral above is evaluated using (1.34), which gives with $Y = -(2iT)^{-1} + G^{-2}$

$$(6.33) \qquad I \ll GL + \Big|\sum_{n\le T'} d(n)n^{-1/2+iq(T)}Y^{-1/2}\exp(-(\log n/T')^2/4Y)\Big| \ll$$

$$\ll GL + G\Big|\sum_{n\le T'} d(n)n^{-1/2+iq(T)}\exp(-\frac{G^2}{4}(\log n/T')^2)\Big|.$$

The presence of the negative exponential factor in (6.33) will make the contribution of many summands negligible. To see this let $n = [T'] - m$ and suppose $m \ge T'G^{-1}L^{(1+\delta)/2}$, where $\delta > 0$ is arbitrary small, but fixed. For these m we have

$$G^2(\log n/T')^2 \gg G^2m^2/(T')^2 \gg L^{1+\delta},$$

and since $\exp(-c_1L^{1+\delta}) < T^{-c_2}$ for any fixed $c_1, c_2 > 0$, we have

$$(6.34) \qquad I \ll GL + G|S|,$$

where S is the "short" exponential sum

$$(6.35) \qquad S = \sum_{T'-T'G^{-1}L^{(1+\delta)/2}\le n\le T'} d(n)n^{-1/2-iT}\exp(-\frac{G^2}{4}(\log n/T')^2).$$

From now on the idea of the proof is to apply Voronoï's summation formula (3.2) to S in (6.35), obtaining eventually (6.25). Since the summation formula (3.2) involves an infinite series whose tails are not easy to estimate, we shall use Lemma 2.3, and instead of (6.35) it will be more suitable to consider the averaged sum

$$(6.36)\quad S' = U^{-1}\int_0^U S(u)\,du,\; S(u) = \sum_{T'-T'G^{-1}L^{(1+\delta)/2}+u\le n\le T'-u} d(n)n^{-1/2-iT}\exp(2\pi i n-\tfrac{G^2}{4}(\log n/T')^2).$$

The factor $e^{2\pi i n} = e(n) = 1$ that is introduced here does not affect the value of the sum, but is inserted to regulate the distribution of saddle points of exponential integrals which arise after Voronoĭ's formula is applied. We shall choose

$$(6.37)\qquad U = G^{1/2},$$

and then trivially $S - S' \ll UT^{-1/2+\varepsilon}$. Now we apply (3.2) to $S(u)$ in (6.36), setting for convenience of notation $M_1 = T' - T'G^{-1}L^{(1+\delta)/2}, M_2 = T' = T/(2\pi)$. Then (3.2) gives

$$(6.38)\quad S(u) = \int_{M_1+u}^{M_2-u} (\log x+2\gamma)x^{-1/2-iT}\exp(2\pi i x - \tfrac{G^2}{4}(\log x/T')^2)\,dx + O(1) +$$

$$+ \sum_{n=1}^{\infty} d(n)\int_{M_1+u}^{M_2-u} x^{-1/2-iT}\exp(2\pi i x - \tfrac{G^2}{4}(\log x/T')^2)\alpha(nx)\,dx,$$

where $\alpha(x)$ is defined by (3.3). We recall the asymptotic formula (3.15), which we write here again as

$$(6.39)\quad \alpha(nx) = -2^{1/2}x^{-1/4}n^{-1/4}\Big\{\sin(4\pi\sqrt{nx}-\pi/4) - (32\pi)^{-1}(nx)^{-1/2}\cos(4\pi\sqrt{nx}-\pi/4)\Big\} + $$
$$+ O(n^{-5/4}x^{-5/4}),$$

noting first that the contribution of the O-term above to the sum in (6.38) is certainly $\ll 1$. The first integral in (6.38) is estimated by (2.5) as

$$\ll M_1^{-1/2}L \max_{M_1\le x\le M_2} (Tx^{-2})^{-1/2} \ll L,$$

and therefore its contribution to S' in (6.36) is again $\ll L$ and so $\ll GL$ in (6.34). To treat the terms containing sines and cosines in (6.39) let N be defined by (6.23) and write the series in (6.38) as

$$(6.40)\qquad \sum_{n=1}^{\infty} d(n) \int_{M_1+\mu}^{M_2-\mu} \ldots \alpha(nx)\,dx = R_1 + R_2 + O(1),$$

where

$$(6.41)\quad R_1 = -2^{1/2} \sum_{n \le (1+\varepsilon)N} d(n) n^{-1/4} \int_{M_1+\mu}^{M_2-\mu} x^{-3/4-iT} \Big\{ \sin(4\pi\sqrt{nx} - \pi/4)$$

$$- (32\pi)^{-1}(xn)^{-1/2} \cos(4\pi\sqrt{nx} - \pi/4) \Big\} \exp(2\pi i x - \frac{G^2}{4}(\log x/T')^2)\,dx,$$

$$(6.42)\quad R_2 = -2^{1/2} \sum_{n > (1+\varepsilon)N} d(n) n^{-1/4} \int_{M_1+\mu}^{M_2-\mu} \ldots,$$

where in (6.42) ... stands for the same terms as in (6.41). The sum R_2 will be estimated as $\ll L$, and this can be at once seen for terms coming from $\cos(4\pi\sqrt{nx}-\pi/4)$. Namely using (2.5) with $f(z) = z - T'\log z \pm 2\sqrt{nz}$ we have

$$\int_{M_1+\mu}^{M_2-\mu} \exp(-\frac{G^2}{4}(\log x/T')^2) x^{-5/4-iT} e(x - T'\log x \pm 2\sqrt{nx})\,dx \ll M_1^{-5/4} \max_{M_1 \le x \le M_2} |f'(x)|^{-1} \ll M_1^{-5/4} (M_1/n)^{1/2},$$

since for $M_1 \le x \le M_2$ and $n > (1+\varepsilon)N$ we have $|f'(x)| \gg (n/x)^{1/2}$, and therefore the cosine terms in (6.42) contribute a total of

$$\sum_{n > (1+\varepsilon)N} d(n) n^{-5/4} . M_1^{-3/4} = o(1).$$

To estimate the contribution of sine terms in (6.42) we shall make use of S', as defined by (6.36). By the properties of the function $\alpha(nx)$ we may integrate termwise, and we are left with the estimation of

$$(6.43)\quad \sum_{n > (1+\varepsilon)N} d(n) n^{-1/4} \Big| U^{-1} \int_0^U \int_{M_1+\mu}^{M_2-\mu} x^{-3/4-iT} \exp(2\pi i x - \frac{G^2}{4}(\log x/T')^2 \pm 4\pi i\sqrt{nx})\,dx\,du \Big|,$$

which will be carried out with the use of Lemma 2.3, where we take

$$f(z) = z \pm 2\sqrt{nz} - T'\log z, \quad g(z) = z^{-3/4} \exp(-\frac{G^2}{4}(\log z/T')^2),$$

$a = M_1, b = M_2$, $G = M_1^{-3/4}$, $\mu \asymp M_1$, $M = (n/M_1)^{1/2}$. This is the only point in the proof where the parameter $U = G^{1/2}$ is needed, and we can easily verify that the hypotheses of Lemma 2.3 are fulfilled. Taking into account that $TG^{-2}L^{1+\delta} \asymp N$ we therefore obtain that the expression given by (6.43) is

$$\ll M_1^{-3/4}\sum_{n>N} d(n)(n^{-1/4}M_1\exp(-A(nT)^{1/2}) + n^{-5/4}M_1U^{-1}) \ll$$

$$M_1^{1/4}U^{-1}N^{-1/4}L \ll G^{1/2}U^{-1}L = L.$$

We have therefore terminated the estimation of R_2 in (6.42) and now we turn to R_1 in (6.41). First we observe that the contribution of terms with $\cos(4\pi\sqrt{nx} - \pi/4)$ is trivially

$$\ll \sum_{n\le(1+\varepsilon)N} d(n)n^{-3/4}\int_{M_1}^{M_2} x^{-5/4}dx \ll N^{1/4}LT^{-1/4} \ll L.$$

Since U is needed not anymore in (6.41) we replace the limits of integration in (6.41) by M_1 and M_2 respectively, making an error which is

$$\ll \sum_{n\le(1+\varepsilon)N} d(n)n^{-1/4}\Big(\Big|\int_{M_1}^{M_1+U}\Big| + \Big|\int_{M_2-U}^{M_2}\Big|\Big) \ll \sum_{n\le(1+\varepsilon)N} d(n)n^{-1/4}UT^{-3/4}$$

$$\ll N^{3/4}LUT^{-3/4} \ll UG^{-3/2}L \ll 1.$$

Thus we have yet to consider

$$i2^{-1/2}\sum_{n\le(1+\varepsilon)N} d(n)n^{-1/4}I_n^{\pm}, \tag{6.45}$$

where

$$I_n^{\pm} = \int_{M_1}^{M_2} x^{-3/4}\exp(-\tfrac{G^2}{4}(\log x/T')^2)\exp(2\pi ix - iT\log x \pm i(4\pi\sqrt{nx} - \pi/4))dx, \tag{6.46}$$

which means that in I_n^+ the + sign is to be taken in $\exp(2\pi ix - \ldots)$, while in I_n^- the minus sign is to be taken.

To estimate $I_n^{\pm}$ we shall apply Theorem 2.2 with $a = M_1, b = M_2$, $k = 1$, $f(z) = f_n^{\pm}(z) = -T'\log z \pm 2(nz)^{1/2}$, $\varphi(z) = z^{-3/4}\exp(-\frac{G^2}{4}(\log z/T')^2)$, $\phi(x) = x^{-3/4}$, $F(x) = T$, $\mu(x) = T$. The conditions of Theorem 2.2 are readily verified, e.g. $N \ll TG^{-2}\log^{1+\delta}T$ implies $|f_n^{\pm}(z)''|^{-1} \ll \mu^2(x)F^{-1}(x)$. The saddle points are the roots of the equation $f_n^{\pm}(x)' = -1$, which is

$$1 - T/(2\pi x) \pm (n/x)^{1/2} = 0, \tag{6.47}$$

and these roots must lie in $[M_1, M_2]$ for the main terms in Theorem 2.2 to exist. Of the two roots of the equation (6.47) only

(6.48) $$x_o = T/(2\pi) + n/2 - (n^2/4 + nT/2\pi)^{1/2}$$

need be considered, since the other root always exceeds M_2, and x_o corresponds to the integral I_n^+. Trivially $x_o \le M_2 = T' = T/(2\pi)$, and $x_o \ge M_1$ holds for $n \le n_o$, where

$$T' + n_o/2 - (n_o^2/4 + n_oT')^{1/2} = T' - B,\ B = T'G^{-1}L^{(1+\delta)/2},$$

hence

$$-n_oT'/(n_o/2 + (n_o^2/4 + n_oT')^{1/2}) = -B.$$

Solving for n_o we obtain

$$n_o = N = B^2/(T' - B),$$

where N is given by (6.23).

The error terms arising from I_n^+ and I_n^- after Theorem 2.2 is applied are treated analogously, and thus only error terms coming from I_n^+ will be considered (no main terms come from I_n^-). Alternatively, one can estimate the sum with I_n^- by applying (2.3). To calculate the main terms coming from the saddle points of I_n^+ note that

$$f_n'(z) = f_n'^+(z) = -T'/z + (n/z)^{1/2},$$

$$f_n''(z) = T'z^{-2} - \tfrac{1}{2}n^{1/2}z^{-3/2} = z^{-3/2}(T'z^{-1/2} - \tfrac{1}{2}n^{1/2}),$$

and in view of $(n/x_o)^{1/2} = T'/x_o - 1$ it follows that

$$f_n''(x_o) = x_o^{-3/2}(T'x_o^{-1/2} - \tfrac{1}{2}n^{1/2}),$$

$$T'x_o^{-1/2} - \tfrac{1}{2}n^{1/2} = T'n^{-1/2}(T'/x_o - 1) - \tfrac{1}{2}n^{1/2} = n^{1/2}(1/4 + T'/n)^{1/2},$$

since rationalizing the right-hand side of (6.48) we obtain

$$T'^2/(nx_o) = T'/n + 1/2 + (1/4 + T'/n)^{1/2}.$$

Hence

$$(f_n''(x_o))^{-1/2} = x_o^{3/4}n^{-1/4}(1/4 + T'/n)^{-1/4},$$

and likewise

$$T'x_o^{-1} = 1 + n/2T' + (n^2/4T'^2 + n/T')^{1/2} = ((n/4T')^{1/2}+(1 + n/4T')^{1/2})^2,$$

which gives

$$\log(T'/x_o) = 2\log((n/4T')^{1/2}+(1 + n/4T')^{1/2}) = 2\,\mathrm{ar\,sinh}((n/4T')^{1/2}).$$

Therefore

$$\varphi(x_o)f_n''(x_o)^{-1/2}e(f(x_o) + kx_o + 1/8) =$$

$$n^{-1/4}(1/4 + T'/n)^{-1/4}\exp(-(\mathrm{Gar}\ \sinh((n/4T')^{1/2}))^2)e(-T'\log x_o + 2(nx_o)^{1/2} + x_o + 1/8),$$

and since $e(-n/2) = e^{n\pi i} = (-1)^n$ it is seen using (6.48) and $(nx_o)^{1/2} = T' - x_o$ that

$$\Big|\sum_{n\le N} d(n)n^{-1/4}\varphi(x_o)f_n''(x_o)^{-1/2}e(f(x_o) + x_o + 1/8)\Big| \le$$

$$\Big|\sum_{n\le N}(-1)^n d(n)n^{-1/2}(1/4 + T'/n)^{-1/4}\exp(if(T,n))\exp(-(\mathrm{Gar}\ \sinh((n/4T')^{1/2}))^2)\Big|,$$

where $f(T,n)$ is given by (6.22), and the last sum is exactly the one that appears in (6.25).

Thus it remains to show that the contribution of error terms of I_n^+ to the sum in (6.45) is $\ll L$. This is analogous to the corresponding proof of the approximate functional equation (4.11) by the use of Voronoï's formula, and the only terms which are non-trivial to estimate are

$$\sum_{n\le(1+\varepsilon)N} d(n)n^{-1/4}T^{-3/4}\Big\{(|f_n'(M_1)+1|+T^{-1/2})^{-1} + (|f_n'(M_2)+1|+T^{-1/2})^{-1}\Big\},$$

since $f_n''(x)\asymp T^{-1}$ for $M_1 \le x \le M_2$. Next $f_n'(M_2) + 1 = (n/T')^{1/2}$, and this gives

$$\sum_{n\le(1+\varepsilon)N} d(n)n^{-1/4}T^{-3/4}(|f_n'(M_2)+1|+T^{-1/2})^{-1} \ll T^{-3/4}\sum_{n\le 2N} d(n)n^{-3/4}T^{1/2} \ll L.$$

The equation $1 + f_n'(M_1) = 0$ has only one solution in n, namely $n = N$, so it is convenient to write $n = [N] + k$. Then

$$1 + f_n'(M_1) = f'_{N+k}(M_1) - f_N'(M_1) \asymp |k|N^{-1/2}T^{-1/2},$$

since $(N+k)^{1/2} - N^{1/2} \asymp |k|N^{-1/2}$ for $|k| \le N/2$. Also for $\varepsilon > 0$ fixed and sufficiently small for $|k| > \frac{\varepsilon}{2}N$ in $f_n'(M_1) + 1$ either the term $1 - T'/M_1 \asymp G^{-1}L^{(1+\delta)/2}$ or the term $(n/M_1)^{-1/2}$ dominates, and in either case we have

$$(|f_n'(M_1)+1| + T^{-1/2})^{-1} \ll \begin{cases} T^{1/2}, & |k| < T^{1/2}G^{-1}L^{(1+\delta)/2} \\ |k|^{-1}TG^{-1}L^{(1+\delta)/2}, & T^{1/2}G^{-1}L^{(1+\delta)/2} \le |k| \le \frac{\varepsilon}{2}N. \\ \max(G, T^{1/2}n^{-1/2}), & |k| > \frac{\varepsilon}{2}N \end{cases}$$

Therefore finally

$$T^{-3/4}\sum_{n\le(1+\varepsilon)N} d(n)n^{-1/4}(|f_n'(M_1)+1|+T^{-1/2})^{-1} \ll$$

$$T^{-1/4}\sum_{|k|\le T^{1/2}G^{-1}L^{(1+\delta)/2}} d(N+k)(N+k)^{-1/4} + T^{1/4+\varepsilon}G^{-1}N^{-1/4} +$$

$$T^{\varepsilon-1/4}\sum_{\frac{\varepsilon}{2}N\le|k|\le\varepsilon N} d(N+k)(N+k)^{-3/4} + GT^{-3/4}\sum_{\frac{\varepsilon}{2}N\le|k|\le\varepsilon N} d(N+k)(N+k)^{-1/4} \ll L,$$

which completes the proof of Theorem 6.2.

In concluding this section let it be recalled that the proof of Theorem 6.2 depended on the use of the approximate functional equation (4.13) for $|\zeta(1/2+it)|^2$, whose proof is not easy. Instead of (4.13) one may use the reflection principle, which is simpler than the approximate functional equation for $\zeta^2(s)$, and obtain a result of the same strength as Theorem 6.2 (the unimportant exponential factors which will appear in the course of the proof may be easily removed by partial summation). Namely from (4.66) with $h = \log^2 T$, $k = 2$, $T - GL \le t \le T + GL$, $s = 1/2 + it$, $M = 4T^2/Y$, $w = u + iv$, $\alpha = 1/2 + \varepsilon$ we have

$$\zeta^2(1/2+it) = \sum_{n\le 2Y} d(n)e^{-(n/Y)^h}n^{-1/2-it} + \chi^2(1/2+it)\sum_{n\le 4T^2/Y} d(n)n^{-1/2+it} +$$

$$+ O(1) - (2\pi i)^{-1}\int_{u=\varepsilon,\,|v|\le h^2} \chi^2(1/2+it+w)\sum_{n\le 4T^2/Y} d(n)n^{-1/2+it+w}Y^w\Gamma(1+w/h)\frac{dw}{w}.$$

Here we choose $Y = 2T$ to equalize the length of the sums and multiply by $\chi^{-1}(1/2+it)$ to obtain

$$\chi^{-1}(1/2+it)\zeta^2(1/2+it) = |\zeta(1/2+it)|^2.$$

From this point on the proof would be quite similar to the one given already for Theorem 6.2. The reflected sum (with $-1/2 + it + w$) will give the same upper bound as the other two, namely (6.20).

§5. The order of the zeta-function in the critical strip

The problem of finding the order of $|\zeta(\sigma+it)|$ in the "critical strip" $0 < \sigma < 1$ is one of the deepest problems of analytic number theory, with many different applications. The present state of knowledge is far from satisfactory, unless one accepts the truth of unproved conjectures like Lindelöf's or Riemann's. Lemma 6.1 and Theorem 6.1 provide at once the means for the estimation of $|\zeta(\sigma+it)|$, $1/2 < \sigma < 1$ fixed. By (6.1) and (6.13) we have

$$|\zeta(\sigma + iT)|^2 \ll 1 + \log T \cdot \int_{T-\log^2 T}^{T+\log^2 T} |\zeta(\sigma - 1/\log T + iv)|^2 dv \ll$$

$$1 + \log T \cdot \int_{T-G}^{T+G} |\zeta(\sigma - 1/\log T + iv)|^2 dv \ll G\log T$$

for

$$G \geq T^{(p+q+1-2\sigma)/(2p+2)}(\log T)^{(2+p)/(1+p)}, \text{ if } 1 + q - p \geq 2\sigma,$$

where (p,q) is an exponent pair . This gives

$$\zeta(\sigma + iT) \ll T^{(p+q+1-2\sigma)/(4p+4)}(\log T)^{(3+2p)/(2p+2)}, \ 1/2 < \sigma < 1, \tag{6.49}$$

if $1 + q - p \geq 2\sigma$, and for $\sigma = 1/2$ we obtain

$$\zeta(1/2 + iT) \ll T^{(p+q)/(4p+4)}(\log T)^{(4+3p)/(2p+2)}. \tag{6.50}$$

If we define for any real θ the function $c(\theta)$ in such a way that

$$\zeta(\theta + iT) \ll T^{c(\theta)+\varepsilon} \tag{6.51}$$

for any $\varepsilon > 0$ and $T \geq T_0(\varepsilon)$, then finding the order of the zeta-function means in fact finding upper bounds for $c(\theta)$. The bound furnished by (6.49) provides fairly good estimates with an adequate choice of (p,q), while (6.50) yields with $(p,q) = (11/30, 16/30)$ the bound $c(1/2) \leq 27/164 = 0.164634\ldots$, which was proved in E.C. Titchmarsh's book [8]. With the theory of exponent pairs good upper bounds for $c(\theta)$ may be derived from the approximate functional equation (4.10) for $\zeta(s)$. Namely choosing $x = y = (t/2\pi)^{1/2}$ in (4.10) we have by partial summation

$$\zeta(\sigma + it) \ll 1 + \sum_N N^{-\sigma} \max_{N<N'\leq 2N} \Big|\sum_{N<n\leq N'} n^{it}\Big| \quad + \tag{6.52}$$

$$+ \sum_N N^{\sigma-1} t^{1/2-\sigma} \max_{N<N'\leq 2N} \left| \sum_{N<n\leq N'} n^{it} \right| ,$$

where N takes $O(\log T)$ values of the form $(t/2\pi)^{1/2} 2^{-j}$, $j = 1,2,\ldots$. Since $N \ll t^{1/2}$ we may use the theory of exponent pairs (see §3 of Chapter 2) to estimate

$$(6.53) \qquad S(N,t) = \sum_{N<n\leq N'} n^{it} = \sum_{N<n\leq N'} \exp(iF(n)),\ F(x) = t\log x.$$

Here $F'(x) = t/x \gg 1$ for $N \leq x \leq 2N$, and also $F^{(k)}(x) \asymp tN^{-k}$ for $k = 1,2,$ $\ldots$. Therefore we obtain

$$(6.54) \qquad S(N,t) \ll (tN^{-1})^p N^q,$$

and from (6.51) and (6.54) we infer that

$$(6.55) \qquad \zeta(\sigma + it) \ll 1 + \sum_N (t^p N^{q-p-\sigma} + t^{1/2-\sigma+p} N^{q-p+\sigma-1}).$$

If further we have

$$(6.56) \qquad \sigma \geq 1/2,\ q - p \geq \sigma,$$

then (6.55) gives

$$\zeta(\sigma + it) \ll 1 + (t^{p+(q-p-\sigma)/2} + t^{1/2-\sigma+p+(q-p+\sigma-1)/2})\log t,$$

or

$$(6.57) \qquad \zeta(\sigma + it) \ll t^{(q+p-\sigma)/2}\log t,\ q - p \geq \sigma,\ \sigma \geq 1/2.$$

In case $\sigma = 1/2$ (when $q - p \geq 1/2$ has to be observed) we get a small improvement of $c(1/2) \leq 27/164$, viz.

$$(6.58) \qquad c(1/2) \leq 0.164510678\ldots,$$

by taking the exponent pair $(p,q) = (\alpha/2 + \varepsilon, 1/2 + \alpha/2 + \varepsilon)$, $\alpha = 0.3290213568\ldots$, and this seems to be the present limit obtainable by the method of exponent pairs in the one-dimensional case. The classical estimates of van der Corput and Hardy-Littlewood (see Titchmarsh [8], Chapter 5) state that for $L = 2^{l-1}$, $l \geq 3$ one has

$$(6.59) \qquad c(\theta) \leq 1/(2L - 2) \qquad \text{for } \theta = 1 - l/(2L - 2),$$

$$(6.60) \qquad c(\theta) \leq 1/L(l + 1) \qquad \text{for } \theta = 1 - l/L.$$

Starting from the exppnent pair $(p,q) = (1/6, 2/3)$ and using Lemma 2.8 (A-process) it is seen by induction on l that $(p,q) = (1/(2L-2), (2L-l-1)/(2L-2))$ is also an exponent pair, so that (6.59) follows from (6.57) and similarly one arrives at (6.60).

From the functional equation (1.3) it follows at once that $c(\theta) = 1/2 - \theta$ for $\theta \le 0$, $c(\theta) = 0$ for $\theta \ge 1$. Since $c(\theta)$ is a non-increasing, convex function of θ it is seen that upper bound for $c(\theta)$ may be obtained in a satisfactory way from (6.57)-(6.60), convexity, and the functional equation for the zeta-function.

As mentioned at the end of Chapter 2, E.C. Titchmarsh developed in the 1930's a powerful two-dimensional method for the estimation of exponential sums. He considered sums of the type $\sum_{(x,y)\in D} e(f(x,y))$, where (x,y) is a point whose coordinates are integers, and which lies in a two-dimensional domain D, while f is a function of two variables possessing at least the partial derivatives of the second order. Several variants of the two-dimensional method have appeared in the past fifty years, and at present the best results seem to be those coming from the method of G. Kolesnik [3], [5], [6]. In some problems, like in the estimation of $c(1/2)$, it is not easy to transform a one-dimensional exponential sum into a two-dimensional sum, although a general procedure is being offered by Lemma 2.6, especially by (2.38). However Theorem 6.2 and Lemma 6.1 provide us with the means of applying two-dimensional techniques at once, since (6.21) contains the divisor function $d(n)$, and therefore $S(x)$ is really a two-dimensional sum. Before obtaining estimates of $c(1/2)$ and $\int_{T-G}^{T+G} |\zeta(1/2+it)|^2 dt$ we shall quote the following result of G. Kolesnik [6].

<u>Lemma 6.3</u>. Let D be the domain $X \le x \le X_1 \le 2X$, $Y \le y \le Y_1 \le 2Y$, $XY = N$, where x and y are positive integers. Then

$$(6.61) \qquad \sum_{(x,y)\in D} e(f(x,y)) \ll N\log N(N^{61/38}F^{-1} + N^{-85/38}F)^{1/8},$$

where $f(x,y) \ll F$, $f_{x^k y^l}(x,y) = C_{k,l} f(x,y)x^{-k}y^{-l} + O(\delta F x^{-k} y^{-l})$ for $(x,y) \in D$, $\delta \ll N^{-1/3}$, $X \ge Y$, and a certain system of equations involving partial derivatives of $f(x,y)$ must be satisfied.

This last condition mentioned above, involving a certain system of equations, is rather technical and lengthy, and therefore is not stated in detail for brevity's sake. For the details of the difficult proof of (6.61) the reader is referred to Kolesnik [6], since a complete proof of this result falls beyond the scope of this text. Having at disposal an estimate such as (6.61) is, one can improve a little (6.19) and also (6.58) by proving

THEOREM 6.3. For $G \geq T^{35/108}$ and $T \geq T_0$

$$(6.62) \qquad \int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt \ll G\log^2 T.$$

In view of Lemma 6.1, the above estimate yields for $G = T^{35/108}$

$$|\zeta(1/2 + iT)|^2 \ll \log T + \log T \int_{-\log^2 T}^{\log^2 T} e^{-|u|} |\zeta(1/2 + iT + iu)|^2 du \ll$$

$$(1 + \int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt)\log T \ll T^{35/108}\log^3 T,$$

giving at once

Corollary 6.1. For $T \geq T_0$

$$(6.63) \qquad \zeta(1/2 + iT) \ll T^{35/216}\log^{3/2} T.$$

This improves (6.58) since $35/216 = 0.162037037\ldots$. The estimate (6.63) is the best of its kind at the moment of writing this text. G. Kolesnik's proof [6] of (6.63) had the weaker T^{ε} instead of $\log^{3/2} T$.

Proof of Theorem 6.3. The interesting range in (6.62) is $G \leq T^{1/3}$, since the larger values of G are covered by the proof of Theorem 6.1. We use Theorem 6.2 with $\delta = 1$, which leads to the estimation of the sum

$$(6.64) \qquad S(x,K,T) = \sum_{K<n\leq K+x} (-1)^n d(n)\exp(if(T,n)),$$

where $f(T,n)$ is given by (6.22). The factor $(-1)^n$, which is present in (6.64), is harmless. Indeed, if n,k,k_1,m,m_1 are positive integers, $0 < a < b$, and $g(n)$ is any arithmetical function, then

$$\sum_{a<n\leq b} (-1)^n d(n)g(n) = \sum_{a<km\leq b} (-1)^{km} g(km) = \sum_{a/2<k_1m\leq b/2} g(2k_1m) +$$

$$+ \sum_{a/2<km_1\leq b/2} g(2km_1) - \sum_{a/4<k_1m_1\leq b/4} g(4k_1m_1) - \sum_{a<(2k_1+1)(2m_1+1)\leq b} g((2k_1+1)(2m_1+1)).$$

But the last sum above is equal to

$$\sum_{a<km\le b} g(km) - \sum_{a/2<k_1m\le b/2} g(2k_1m) - \sum_{a/2<km_1\le b/2} g(2km_1) + \sum_{a/4<k_1m_1\le b/4} g(4k_1m_1),$$

which shows that the estimation of (6.64) reduces to the estimation of several sums of the type

$$(6.65)\qquad S_1(x,K,T) = \sum_{K/C<mn\le(K+x)/C} \exp(if(T,Cmn)),$$

where $C = 1,2$ or 4. All these sums are estimated analogously, and thus henceforth we shall assume that $C = 1$. Writing

$$g(z) = \operatorname{ar\,sinh} z + z(z^2+1)^{1/2}$$

it is seen that $g'(z) = 2(z^2+1)^{1/2}$, hence for $z < 1$

$$(6.66)\qquad g(z) = \sum_{m=1}^{\infty} a_m z^{2m-1},\quad a_m = \frac{2}{2m-1}\binom{1/2}{m-1},$$

and thus $a_1 = 2$ and $|a_m| \le 1/(2m+1)$. for $m \ge 2$. Therefore with some suitable constants $b_1, b_2, \ldots$ we may write

$$(6.67)\qquad f(T,u) = \sum_{j=1}^{\infty} b_j T^{3/2-j} u^{j-1/2} = F(T,u) + \sum_{j=3}^{\infty} b_j T^{3/2-j} u^{j-1/2},$$

$$(6.68)\qquad F(T,u) = b_1 T^{1/2} u^{1/2} + b_2 T^{-1/2} u^{3/2},$$

and the function F will be easier to estimate by Kolesnik's Lemma 6.3 than f itself. If we suppose that $G \ge T^{1/4}$, then for $\delta < 1$ in Theorem 6.2 we have that $K \le TG^{-2}\log^2 T$, and hence

$$(6.69)\qquad \sum_{K<mn\le K+x} (\exp(if(T,mn)) - \exp(iF(T,mn))) \ll \sum_{K<n\le K+x} d(n)|f(T,n) - F(T,n)|$$

$$\ll T^{\varepsilon} K \max_{K<n\le K+x} |f(T,n) - F(T,n)| \ll T^{\varepsilon} K^{7/2} T^{-3/2} \ll T^{\varepsilon} K^{1/2}.$$

Therefore for $G \ge T^{1/4}$ in Theorem 6.2 we have reduced the problem to the estimation of the sum

$$(6.70)\qquad S_2(x,K,T) = \sum_{K<mn\le K+x} \exp(iF(T,mn)),\quad F(T,u) = b_1 T^{1/2} u^{1/2} + b_2 T^{-1/2} u^{3/2}.$$

To estimate $S_2(x,K,T)$ we apply Lemma 6.3, taking $f(x,y) = F(x,y)$ (here F refers to (6.68)), $F = (TK)^{1/2}$, $N = K$, and dividing the domain $K < mn \le K + x$ into $O(\log T)$ subdomains of the form $X \le x \le X_1 \le 2X$, $Y \le y \le Y_1 \le 2Y$, $X \ge Y$. It may be readily verified that the conditions of Lemma 6.3 hold, and we obtain

$$S_2(x,K,T) \ll \log^2 T(K^{173/152}T^{-1/16} + K^{119/152}T^{1/16}). \tag{6.71}$$

Further observe that for any fixed $C > 0$ we have

$$\sum_{K=2^k\le TG^{-2}\log^2 T} K^C\exp(-G^2K/T) = \sum_{K=2^k\le TG^{-2}} + \sum_{K=2^k>TG^{-2}} \ll (TG^{-2})^C. \tag{6.72}$$

Therefore for $G \ge T^{35/108}$ it follows from Theorem 6.2 and the above estimates that

$$\int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt \ll$$

$$G\log^2 T(1 + \sum_{K=2^k\le TG^{-2}\log^2 T} (K^{135/152}T^{-5/16} + K^{81/152}T^{-3/16} + K^{1/4}T^{\varepsilon-1/4})e^{-G^2K/T})$$

$$\ll G\log^2 T(1 + T^{175/304}G^{-540/304} + T^{105/304}G^{-324/304} + T^{\varepsilon}G^{-1/2}) \ll G\log^2 T,$$

which completes the proof of Theorem 6.3.

§6. Third and fourth power moments in short intervals

Having developed a method based on the use of Fourier coefficients of cusp forms and Kloosterman sums, H. Iwaniec [2] recently obtained a deep estimate for the fourth power moment in short intervals. His Theorem 4 states the following: If $T \ge 2$, $T^{1/2} < G \le T$ and $T \le t_1 < t_2 < \dots < t_R \le 2T$, $t_{r+1} - t_r \ge G$ for $r = 1,\dots,R$, then

$$\sum_{r\le R} \int_{t_r}^{t_r+G} |\zeta(1/2 + it)|^4 dt \ll T^{\varepsilon}(RG + R^{1/2}G^{-1/2}T). \tag{6.73}$$

The proof of this result is too complex to be included here, and (6.73) is used in this text only for the derivation of the approximate functional equation for $\zeta^k(s)$ $(k > 2)$ in Chapter 4, where the special case

$$\int_{T-G}^{T+G} |\zeta(1/2 + it)|^4 dt \ll T^{\varepsilon}G, \quad G \ge T^{2/3} \tag{6.74}$$

of (6.73) is needed. It was mentioned by Iwaniec [2] that (6.73) may be used for the proof of the twelfth power moment estimate (7.15) of Heath-Brown [1], which will be proved here with the aid of Theorem 6.2. It may be also remarked that in the case of the twelfth power moment Iwaniec's method yields T^{ε} instead of $\log^{17}T$ in (7.15). As an application of (6.73) for R = 1 note that for $G > T^{1/2}$ we have by Theorem 6.2 and the Cauchy-Schwarz inequality

$$(6.75)\qquad \int_{T-G}^{T+G} |\zeta(1/2+it)|^3 dt \le \Big(\int_{T-G}^{T+G} |\zeta(1/2+it)|^2 dt\Big)^{1/2}\Big(\int_{T-G}^{T+G} |\zeta(1/2+it)|^4 dt\Big)^{1/2} \ll$$

$$T^{\varepsilon}G^{1/2}(G + TG^{-1/2})^{1/2} \ll T^{\varepsilon}(G + T^{1/2}G^{1/4}),\quad G > T^{1/2},$$

which is used in Chapter 4 in the proof of the approximate functional equation for $\zeta^3(s)$ (with $G = 2T^{1/2}$).

Other interesting mean values involving the zeta-function were investigated by H. Iwaniec [1] and Deshouillers-Iwaniec [1]. A natural way to attack the sixth power moment for the zeta-function ($\int_0^T |\zeta(1/2+it)|^6 dt \ll T^{1+\varepsilon}$) is to try to prove

$$(6.76)\qquad \int_0^T |\zeta(1/2+it)|^4 \Big|\sum_{n\le N} a_n n^{-it}\Big|^2 dt \ll T^{1+\varepsilon}\sum_{n\le N} |a_n|^2$$

for $N \ll T^{1/2}$, where $a_1,\dots,a_N$ are arbitrary complex numbers, since by the approximate functional equation (4.10) (or the reflection principle) it is seen that $\zeta(1/2+it)$ may be majorized by two Dirichlet polynomials of length $\ll T^{1/2}$. Proving (6.76) for the range $N \ll T^{1/2}$ seems to be out of reach at present, but using intricate techniques involving Kloosterman sums, H. Iwaniec [1] obtained (6.76) for the range $N \le T^{1/10}$, while J.-M. Deshouillers and H. Iwaniec [1] improved this to $N \le T^{1/5}$. They also mention that under the truth of a certain conjecture involving the lower bound of eigenvalues of the non-euclidean Laplacian of Hecke congruence subgroups, their method would give $N \le T^{1/4}$.

N O T E S

The proof of Theorem 6.1 is due to the author and has not appeared in print before. An interesting problem seems to be the estimation of $R(k,\sigma;T)$, where

$$(6.77)\qquad \int_1^T |\zeta(\sigma + it)|^{2k} dt = T\sum_{n=1}^{\infty} d_k^2(n)n^{-2\sigma} + R(k,\sigma;T),$$

$k \geq 1$ is a fixed integer, $1/2 < \sigma < 1$ is fixed. This problem will be investigated in more detail in §4 of Chapter 7, while the analogue of (6.77) when $k = 1$, $\sigma = 1/2$ is one of the main topics of this text and will be extensively treated in Chapter 11.

In [9] D.R. Heath-Brown uses an inequality due to P.X. Gallagher (Lemma 1.10 of H.L. Montgomery [2]) to obtain an estimate very similar to Lemma 6.2 and proves (6.19) for the range $G \geq T^{1/3}$.

The proof of Theorem 6.2 is new, and is based on M. Jutila's paper [6] and the exponential averaging technique which gives (6.35). Indeed it would have been shorter to apply Jutila's Theorem 1 of [6] to (4.13) and then to integrate the resulting expression, but the proof given here is self-contained. Also several details in the proof are simpler than the corresponding ones in Jutila's proof, since we are dealing here with a special Dirichlet polynomial, while Jutila considers the general case of transforming $S_1(M_1,M_2;t) = \sum_{M_1 \leq n \leq M_2} d(n)n^{-1/2-it}$ by Voronoï's summation formula. This approach, based on the use of Voronoï's formula, seems very natural and the aforementioned paper of Jutila contains generalizations to other Dirichlet polynomials (e.g. whose coefficients are generated by certain cusp forms).

Theorem 6.3 and its Corollary are given in the author's paper [2]. The final version of Kolesnik's estimate (6.61), as published in his paper [6], had $N^{1+\varepsilon}(N^{61/38}F^{-1} + \ldots$ instead of $N\log N(N^{61/38}F^{-1} + \ldots$, but the slightly sharper version used in the text may be obtained by refining his argument a little.

Theorem 6.3 provides the best-known order estimate for $|\zeta(1/2+iT)|$, while Theorem 6.2 serves as a basis for higher power moments which will be discussed in Chapter 7. It is seen from the proof of Theorem 6.2 that in fact one essentially obtains (6.25) without the absolute value sign and (as remarked in detail in Notes of Chapter 11) it is thus unnecessary to use the Halász-Montgomery inequality in Theorem 7.1. Instead one may proceed directly with the Cauchy-Schwarz inequality, simplifying the proof of Theorem 7.1.

Let $c = c(1/2)$ be the constant for which $\zeta(1/2+it) \ll |t|^{c+\varepsilon}$. The estimates of c have slowly evolved from the first significant exponent 1/6 to today's sharpest 35/216, and though the gain over all these years is just 1/216, nevertheless the improvements of the value of c reflect in a certain sense the constant development of modern analytic number theory. Various values of c are given below with a due reference:

$c = 1/6 = 0.166\ldots$	G.H. Hardy and J.E. Littlewood [2], 1921
$c = 163/988 = 0.1649797\ldots$	A. Walfisz [1], 1924
$c = 27/164 = 0.1646341\ldots$	E.C. Titchmarsh [1], 1931
$c = 229/1392 = 0.1645114\ldots$	E. Phillips [1], 1933
$c = 19/116 = 0.1637931\ldots$	E.C. Titchmarsh [6], 1942
$c = 15/92 = 0.1630434\ldots$	S.H. Min [1], 1949
$c = 6/37 = 0.1621621\ldots$	W. Haneke [1], 1963 and Chen Jing-run [1], 1965
$c = 173/1067 = 0.1621368\ldots$	G. Kolesnik [3], 1973
$c = 35/216 = 0.162037037\ldots$	G. Kolesnik [6], 1982.

Order results for $\zeta(s)$ given in this chapter involve upper bounds. Concerning results about lower bounds one may mention the result of R. Balasubramanian and K. Ramachandra (see Ramachandra [6]) that

$$\max_{T\le t\le T+H} |\zeta(1/2 + it)| > \exp\left(\frac{3}{4}\left(\frac{\log H}{\log\log H}\right)^{1/2}\right), \quad 10 \le (\log T)^{0.001} \le H \le T,$$

while for $1/2 < \sigma < 1$ fixed H.L. Montgomery [4] showed that

$$\log |\zeta(\sigma + it)| = \Omega_+(\log^{1-\sigma} t(\log\log t)^{-\sigma})$$

holds for $t > 0$. Also M. Jutila has kindly informed me that in a yet unpublished manuscript he has proved that there exist positive constants a_1, a_2 and a_3 such that for $T \ge 10$

$$\exp(a_1(\log\log T)^{1/2}) \le |\zeta(1/2 + it)| \le \exp(a_2(\log\log T)^{1/2})$$

on a subset of measure at least a_3T of the interval $[0,T]$.

To assess the strength of Iwaniec's estimate (6.74), note that by Lemma 6.1 with $k = 4$

$$|\zeta(1/2 + iT)|^4 \ll \log T\left(1 + \int_{T-G}^{T+G} |\zeta(1/2 + it)|^4 dt\right) \ll GT^{\varepsilon}$$

for $G \ge T^{2/3}$, hence for $G = T^{2/3}$ one obtains $\zeta(1/2 + iT) \ll T^{1/6+\varepsilon}$, which is the classical result of Hardy and Littlewood [2].

Aleksandar Ivić

TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 7

HIGHER POWER MOMENTS

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§1. Introduction

In this chapter we focus our attention on higher power moments (i.e. higher than the fourth, which was discussed in Chapter 5) for $1/2 \le \sigma < 1$ fixed. As was the case with mean value estimates in Chapter 6, we shall distinguish between the cases $\sigma = 1/2$ and $1/2 < \sigma < 1$, and since our main concern will be upper bounds, it seems appropriate to define $M(A)$ $(A \ge 1)$ for any fixed $A \ge 4$ as the number for which

$$\int_1^T |\zeta(1/2 + it)|^A dt \ll T^{M(A)+\varepsilon} \tag{7.1}$$

for any $\varepsilon > 0$. Similarly for $1/2 < \sigma < 1$ fixed we define $m(\sigma)$ as the number for which

$$\int_1^T |\zeta(\sigma + it)|^{m(\sigma)} dt \ll T^{1+\varepsilon} \tag{7.2}$$

for any $\varepsilon > 0$, and naturally we seek upper bounds for $M(A)$ and lower bounds for $m(\sigma)$. This difference between the definitions of $M(A)$ and $m(\sigma)$ seems in place, since $M(A) = 1$ is not known to hold for any $A > 4$ at the moment of writing of this text, while for any fixed $1/2 < \sigma < 1$ it is possible to find a number $m(\sigma) > 4$ such that (7.2) holds (see E.C. Titchmarsh [8], Chapter 7). Results given by Chapter 7 of Titchmarsh [8] will be however substantially improved here, and the results of this chapter will be used for zero-density theorems of Chapter 9 and for divisor problems in Chapter 10. One of the main instruments in our study of (7.1) and (7.2) will be Theorem 6.2, and moreover in view of Lemma 6.1 the estimation of the integrals appearing in (7.1) and (7.2) is essentially equivalent to the estimation of discrete sums of the type

$$\sum_{r \le R} |\zeta(\sigma + it_r)|^B, \quad 1/2 \le \sigma < 1, \tag{7.3}$$

where $B = A$ or $B = m(\sigma)$ and $t_1, \ldots, t_R$ are well-spaced real numbers in the sense that $|t_r| \le T$ and $|t_r - t_s| \ge 1$ for $r \ne s \le R$.

Finally we turn our attention in §4 to asymptotic formulas for $\int_1^T |\zeta(\sigma+it)|^{2k} dt$ (k a fixed integer), and not only to upper bounds of the type (7.2).

§2. Power moments for $\sigma = 1/2$

In this section we shall suppose that $t_1,\dots,t_R$ form a ~~increasin~~ sequence of real numbers which satisfy

$$(7.4)\qquad |t_r| \le T \text{ for } r = 1,2,\dots,R;\ |t_r - t_s| \ge 1 \text{ for } 1 \le r \ne s \le R$$

and

$$(7.5)\qquad |\zeta(1/2 + it_r)| \ge V > 0, \quad (r = 1,2,\dots,R).$$

Our aim is to derive upper bounds for $R = R(V)$, which will eventually lead to estimates of the type (7.1). As an auxiliary reault which is analogous to the fourth power moment, it may be noted that

$$(7.6)\qquad R \ll TV^{-4}\log^5 T,$$

and this will turn out to be the best available bound for R when V is small, as will be precisely seen by comparing (7.6) with later results of this chapter. To see that (7.6) holds it is sufficient to suppose that $T/2 \le t_r \le T$, and then to replace T by $T/2, T/2^2$ etc. and to sum all the results. From the reflection principle estimate (4.66) with $k = 2$, $h = \log^2 T$, $Y = M = 2T$, $s = 1/2 + it_r$, $\alpha = 3/4$ we obtain

$$(7.7)\qquad RV^4 \ll \sum_{r\le R}|\zeta(1/2+it_r)|^4 \ll \sum_{r\le R}\Big|\sum_{n\le 2T} d(n)e^{-(n/2T)^h} n^{-1/2-it_r}\Big|^2 +$$

$$+ \sum_{r\le R}\Big|\sum_{n\le 2T} d(n)n^{-1/2-it_r}\Big|^2 + R +$$

$$+ T^{-1/2}\Big(\max_{|v|\le h^2}\sum_{r\le R}\Big|\sum_{n\le 2T} d(n)n^{-1/4-it_r-iv}\Big|^2\Big)\Big(\int_{-h^2}^{h^2} e^{-|v|h^{-1}}|1/4 + iv|^{-1}dv\Big)^2,$$

where the Cauchy-Schwarz inequality was used, (1.32) and (4.4). The integral above is clearly

$$\ll 1 + \int_1^{h^2} v^{-1}dv \ll \log\log T.$$

The sums over $r \le R$ are estimated by the mean value Theorem 5.3, where one uses (5.26). We obtain

$$RV^4 \ll T\log^5 T + R + T^{-1/2}(\log\log T)^2\Big(T\sum_{n\le 2T} d^2(n)n^{-1/2} + \sum_{n\le 2T} d^2(n)n^{1/2}\Big)\log T$$

$$\ll T\log^5 T,$$

if $V \geq \log T$, and (7.6) follows. For $V < \log T$ (7.6) follows again from the trivial estimate $R \leq T$.

We pass now to the main result of this section, which is

THEOREM 7.1. Let (p,q) be any exponent pair with $p > 0$, and let $t_1 < \ldots < t_R$ satisfy (7.4) and (7.5). Then

$$R \ll TV^{-6}\log^8 T + T^{(p+q)/p}V^{-2(1+2p+2q)/p}(\log T)^{(3+6p+4q)/p}. \tag{7.8}$$

For special choices of the exponent pair (p,q) we obtain from (7.8)

Corollary 7.1. Under the hypotheses of Theorem 7.1 we have

$$R \ll TV^{-6}\log^8 T + T^{29/13}V^{-178/13}\log^{235/13}T, \tag{7.9}$$

$$R \ll TV^{-6}\log^8 T + T^{5/2}V^{-31/2}\log^{81/4}T, \tag{7.10}$$

$$R \ll TV^{-6}\log^8 T + T^{3}V^{-19}\log^{49/2}T, \tag{7.11}$$

$$R \ll TV^{-6}\log^8 T + T^{4}V^{-128/5}\log^{162/5}T, \tag{7.12}$$

$$R \ll TV^{-6}\log^8 T + T^{15/4}V^{-24}\log^{61/2}T. \tag{7.13}$$

The exponent pair $(p,q) = (1/2,1/2)$ in Theorem 7.1 leads to an important result, proved first by D.R. Heath-Brown [1]. This is

Corollary 7.2. Under the hypotheses of Theorem 7.1 we have

$$R \ll T^2V^{-12}\log^{16}T. \tag{7.14}$$

From this result it follows that $M(12) \leq 2$, or more precisely

$$\int_1^T |\zeta(1/2 + it)|^{12}dt \ll T^2\log^{17}T. \tag{7.15}$$

Before the proof of Theorem 7.1 we shall give a lemma providing estimates for moduli of $S(x,K,t_r)$ over well-spaced points t_r, where $S(x,K,T)$ is defined by (6.21). The result is contained in

Lemma 7.1. Let A be a set of real numbers t_r such that $T/2 \leq t_r \leq T$ and $\log^2 T \leq G \leq |t_r - t_s| \leq J$ for $r \neq s$. If $|A|$ denotes the cardinality of A, then for $K \leq T/\log T$, $T \geq T_0$ and any exponent pair (p,q) we have

$$\sum_{t_r \in A} |S(x,K,t_r)| \ll \Big\{(K+K^{3/4}T^{1/4}G^{-1/2}\log^{1/2}T)|A|^{1/2} + J^{p/2}T^{-p/4}|A|K^{(2q-p+2)/4}\Big\}\log^{3/2}T. \tag{7.16}$$

Proof of Lemma 7.1. We start from (1.35), and choose $\xi = \{\xi_n\}_{n=1}^{\infty}$ with $\xi_n = (-1)^n d(n)$ for $K \le n \le K + x$ and zero otherwise, $\varphi_r = \{\varphi_{r,n}\}_{n=1}^{\infty}$, with $\varphi_{r,n} = \exp(if(t_r,n))$ for $K \le n \le 2K$ and zero otherwise, where $f(t_r,n)$ is defined by (6.22). Then by (1.39) we have uniformly for $x \le K$

$$\|\xi\|^2 = \sum_{K<n\le K+x} d^2(n) \ll K\log^3 T,$$

$$(7.17) \qquad \sum_{t_r \in A} |S(x,K,t_r)| \ll K^{1/2}\log^{3/2}T\Big\{\sum_{t_r,t_s\in A}\Big|\sum_{K<n\le 2K}\exp(if(t_r,n)-if(t_s,n))\Big|\Big\}^{1/2}$$

The inner sum on the right-hand side of (7.17) is $O(K)$ if $r = s$, and if $r \ne s$ we shall use the theory of exponent pairs (§3 of Chapter 2) to estimate

$$(7.18) \qquad S = \sum_{K<n\le 2K}\exp(if(n)),\ f(u) = f(t_r,u) - f(t_s,u),\ r \ne s.$$

Defining $g(z) = \operatorname{ar\,sinh} z + z(z^2 + 1)^{1/2}$ we recall that (6.66) holds and since

$$f(u) = 2t_r g((\pi u/2t_r)^{1/2}) - 2t_s g((\pi u/2t_s)^{1/2}),\ r \ne s,$$

it is seen therefore for r,s fixed that for $K \le u \le 2K$ and $j = 1,2,\ldots$

$$f^{(j)}(u) \asymp |t_r - t_s|K^{1/2-j}T^{-1/2},$$

where we used (6.66), the mean value theorem and the condition $K < T/\log T$. This implies that if $F = |t_r - t_s|K^{-1/2}T^{-1/2} \gg 1$ we may use the theory of exponent pairs to estimate (7.18), and if this condition is not satisfied we use Lemma 2.1 and Lemma 2.5 to obtain in any case

$$(7.19) \qquad S = \sum_{K<n\le 2K}\exp(if(n)) \ll F^pK^q + \max_{K\le u\le 2K}|f'(u)|^{-1} \ll$$

$$J^pT^{-p/2}K^{q-p/2} + (KT)^{1/2}|t_r - t_s|^{-1}.$$

The spacing condition imposed on the t_r's gives

$$\sum_{t_r,t_s\in A, r\ne s}|t_r - t_s|^{-1} \ll G^{-1}|A|\sum_{n\le A} n^{-1} \ll G^{-1}|A|\log T,$$

and thus substituting (7.18) and (7.19) in (7.17) we obtain (7.16).

<u>Proof of Theorem 7.1</u>. Having at our disposal Theorem 6.2 and Lemma 7.1 it will be a fairly simple matter to derive Theorem 7.1. Let first A_3 denote a set of points t_r satisfying the spacing condition (7.4) but with $2T/3 \le t_r \le 5T/6$. We shall divide the interval $[2T/3, 5T/6]$ into N subintervals of length at most $J = T/(6N)$, and we shall denote by $A_{1,k}$ $(k = 1,\dots,N)$ the set of points in the k-th of these intervals. Then the points of each $A_{1,k}$ lie in an interval $[T_0, T_0 + J]$ for some T_0 which satisfies $2T/3 \le T_0 \le 5T/6 - J$. We shall estimate first $A_{1,k}$ by taking

$$(7.20) \qquad BG\log^2 T = V^2$$

for some suitable $B > 0$ and defining

$$A'_{1,k} = A'_{1,k}(\tau) = A_{1,k} \cap [\tau - G/2, \tau + G/2]$$

for $7T/12 \le \tau \le 11T/12$. By (6.63) the relevant range for V in Theorem 7.1 is $V \ll T^{35/216}\log^{3/2}T$, hence $G \le T^{1/3}$, which will enable us to use Theorem 6,2, where one requires $T^{\varepsilon} \le G \le T^{1/2-\varepsilon}$. By Lemma 6.1

$$\zeta(1/2 + it_r) \ll \log^{1/2}T$$

or

$$(7.21) \qquad |\zeta(1/2 + it_r)|^2 \ll \log T \cdot \int_{-\log^2 t_r}^{\log^2 t_r} e^{-|u|}|\zeta(1/2 + iu + it_r)|^2 du.$$

We may suppose that $V \ge T^{\varepsilon}$, for otherwise the trivial $R \le T$ is better than the second term in (7.8) for $\varepsilon < 1/12$, and henceforth we suppose that (7.21) holds. Summation of (7.21) over $t_r \in A'_{1,k}$ gives for some absolute $C_1 > 0$

$$(7.22) \qquad |A'_{1,k}|V^2 \le C_1 \log T \cdot \int_{\tau-G}^{\tau+G} |\zeta(1/2 + it)|^2 \sum_{t_r \in A'_{1,k}} e^{-|t-t_r|} dt,$$

provided that

$$[t_r - \log^2 t_r,\ t_r + \log^2 t_r] \subseteq [\tau - G, \tau + G]$$

for $t_r \in [\tau - G/2, \tau + G/2]$, which is certainly satisfied for $V \ge T^{\varepsilon}$. The spacing condition $|t_r - t_s| \ge 1$ $(r \ne s)$ implies that the sum in (7.22) is bounded, and for the integral in (7.22) we use Theorem 6.2 with $\delta = 1$, recalling that for our range of τ we may use (6.24), which gives

$$|A'_{1,k}| BG\log^2 T \le C_2 G\log^2 T +$$

(7.23)

$$C_2 G\log T \sum_{T^{1/3} \le K=2^k \le TG^{-2}\log^2 T} (TK)^{-1/4} e^{-G^2K/(2T)} \left(|S(K,K,\sigma)| + K^{-1}\int_0^K |S(x,K,\sigma)|dx\right).$$

Choosing $B = 2C_2$ the above formula simplifies to

$$(7.24)\quad |A'_{1,k}| \ll \log^{-1} T \sum_K (TK)^{-1/4} e^{-G^2K/(2T)} \left(|S(K,K,\sigma)| + K^{-1}\int_0^K |S(x,K,\sigma)|dx\right).$$

Let now $A_{2,k}$ denote the set of numbers $\sigma = T_0 + G/2 + nG$ such that $A'_{1,k}(\sigma) \ne \emptyset$ and n is such an integer for which $T_0 \le \sigma \le T_0 + J + G/2$. If σ_r and σ_s are two different elements of $A_{2,k}$ we have $G \le |\sigma_r - \sigma_s| \le J$, and we may apply Lemma 7.1 to obtain

$$(7.25)\quad \sum_{\sigma\in A_{2,k}} |A'_{1,k}(\sigma)| \ll \log^{1/2} T \sum_K (TK)^{-1/4} e^{-G^2K/(2T)} \Big\{ (K + K^{3/4}T^{1/4}G^{-1/2}\log T)|A_{2,k}|^{1/2}$$

$$+ |A_{2,k}| J^{p/2} T^{-p/4} K^{(2q-p+2)/4} \Big\}.$$

Using (6.72), the obvious inequalities

$$(7.26)\quad |A_{1,k}| \le \sum_{\sigma\in A_{2,k}} |A'_{1,k}(\sigma)|, \quad |A_{2,k}| \le \sum_{\sigma\in A_{2,k}} |A'_{1,k}(\sigma)|$$

and summing over $K = 2^k$ it follows that

$$\sum_{\sigma\in A_{2,k}} |A'_{1,k}(\sigma)| \ll T^{-1/2}\log T \sum_K K^{3/2} e^{-G^2K/T} + G^{-1}\log^2 T \sum_K K e^{-G^2K/T} +$$

$$(7.27)\quad J^{p/2} T^{-(p+1)/4} \log^{1/2} T \cdot |A_{2,k}| \sum_K K^{(2q-p+1)/4} e^{-G^2K/(2T)} \ll$$

$$TG^{-3}\log^2 T + |A_{2,k}| J^{p/2} G^{(p-1-2q)/2} T^{(q-p)/2} \log^{1/2} T.$$

Therefore using (7.26) we obtain

$$(7.28)\quad |A_{1,k}| \le \sum_{\sigma\in A_{2,k}} |A'_{1,k}(\sigma)| \ll TG^{-3}\log^2 T,$$

provided that for some suitable $C_3 > 0$ we have

(7.29) $$J \le C_3 G^{(2q-p+1)/p} T^{(p-q)/p} \log^{-1/p} T.$$

Now we choose N in such a way that

(7.30) $$J = T/(6N) \le C_3 T^{(p-q)/p} G^{(2q-p+1)/p} \log^{-1/p} T < T/(6N-6).$$

This gives

(7.31) $$N \ll 1 + T^{q/p} G^{-(2q-p+1)/p} \log^{1/p} T$$

and

(7.32) $$A_3 = \sum_{k \le N} |A_{1,k}| \ll NTG^{-3} \log^2 T \ll$$

$$TV^{-6} \log^8 T + T^{(p+q)/p} V^{-2(1+2p+2q)/p} (\log T)^{(3+6p+4q)/p}$$

if $G \le J$. This condition is certainly satisfied for

(7.33) $$G \le C_4 T^{-(q-p)/p} G^{(2q-p+1)/p} \log^{-1/p} T$$

or in view of (7.20) for

(7.34) $$V > T_1 = C_5 T^{(q-p)/(2+4q-4p)} (\log T)^{(3-4p+4q)/(2+4q-4p)}$$

where $C_4, C_5 > 0$. Summing over intervals of the form $[T(5/4)^{-j-1}, T(5/4)^{-j}]$ it is seen from (7.32) that Theorem 7.1 follows if (7.34) is satisfied. If (7.34) does not hold, then (7.8) follows from (7.6), since

$$R \ll TV^{-4} \log^5 T \ll T^{(p+q)/p} V^{-2(1+2p+2q)/p} (\log T)^{(3+6p+4q)/p}$$

for $V < T^{q/(2+4q)} \log^{C_6} T = T_2$. But for T_1 given by (7.34) we have $T_1 < T_2$ for any fixed $C_6 > 0, p > 0$ and T sufficiently large, which completes the proof of Theorem 7.1.

Corollary 7.1 follows from Theorem 7.1 with exponent pairs (13/31,16/31), (4/11,6/11),(2/7,4/7),(5/24,15/24),(4/18,11/18) respectively while Corollary 7.2 follows with the exponent pair (1/2,1/2). If we choose (p,q) = (1/6,2/3) in (7.8) then we obtain

(7.35) $$R \ll T^5 V^{-32} \log^4 T, \quad V \le T^{2/13} \log^{16/13} T,$$

thereby improving the range for which the corresponding estimate (8) of Theorem 2 of Heath-Brown [1] holds. Moreover (7.13) gives

(7.36) $$R \ll \begin{cases} TV^{-6} \log^8 T & V \ge T^{11/72} \log^{5/4} T, \\ T^{15/4} V^{-24} \log^{61/2} T & V \le T^{11/72} \log^{5/4} T, \end{cases}$$

where $11/72 = 0.1527\ldots$, and therefore the first estimate in (7.36) improves (9) of Heath-Brown [1], where one had the range $V \geq T^{2/13}\log^{6/5}T$, and $2/13 = 0.153846 > \frac{11}{72}$. Thus the sixth power estimate $R \ll TV^{-6}\log^8 T$ holds for relatively large values of V, and from (7.8) one may crudely say that in a certain sense either $M(4) = 1$ or $M((2+4p+4q)/p) \leq (p+q)/p$ holds, which will be used in Chapter 9 for zero-density estimates.

Theorem 7.1 provides the means for obtaining power moment estimates when $\sigma = 1/2$, and we shall prove

<u>THEOREM 7.2.</u> If $M(A)$ is defined by (7.1), then

$$(7.37)\qquad M(A) \leq \begin{cases} 1 + (A-4)/8, & 4 \leq A \leq 12, \\ 2 + 3(A-12)/22, & 12 \leq A \leq 178/13 = 13.6923076\ldots \\ 1 + 35(A-6)/216, & A \geq 178/13. \end{cases}$$

Here A is a fixed number which does not have to be an integer. The first of the estimates in (7.37) is implicit in Heath-Brown [1], while the last one is an improvement of his estimate

$$\int_1^T |\zeta(1/2 + it)|^k dt \ll T^{1+173(k-6)/1067}\log^{2k}T, \quad k \geq 15.$$

<u>Proof of Theorem 7.2.</u> As remarked in §1 it will be sufficient to prove the discrete estimate

$$(7.38)\qquad \sum_{r \leq R} |\zeta(1/2 + it_r)|^A \ll T^{M(A)+\varepsilon},$$

where the t_r's satisfy (7.4). To see this define t_r by

$$(7.39)\qquad |\zeta(1/2 + it_r)| = \max_{r \leq t \leq r+1} |\zeta(1/2 + it)|,$$

where r is an integer and the maximum exists by continuity of $|\zeta(1/2 + it)|$ as a function of the real variable t. Then we have

$$(7.40)\qquad I = \int_1^T |\zeta(1/2 + it)|^A dt \ll \sum_{1 \leq r \leq T} |\zeta(1/2 + it_r)|^A,$$

and to obtain the condition $|t_r - t_s| \geq 1$ for $r \neq s$, we consider separately t_{2m} and t_{2m+1}, so that I in (7.40) is majorized by two sums of the form (7.3) with

$r = 1,2,\ldots,R$, $R \le T$, and the t_r's satisfy (7.4). Each $|\zeta(1/2 + it_r)|$ satisfies then

(7.41) $$V \le |\zeta(1/2 + it_r)| < 2V$$

for some $0 < V = 2^k \ll T^{35/216}\log^{3/2}T$, and we define R_V as the number of t_r's appearing in (7.40) which satisfy (7.4) and (7.41).

Suppose now that $4 \le A \le 12$. Then by (7.6) and (7.14) we may use $R \ll TV^{-4}\log^5 T$ for $V \le T^{1/8}\log^{11/8}T$ and $R \ll T^2V^{-12}\log^{16}T$ for $V > T^{1/8}\log^{11/8}T$. Therefore

$$\sum_{r\le R} |\zeta(1/2 + it_r)|^A \ll \sum_{V=2^k\le T^{1/8}\log^{11/8}T} R_V V^A + \sum_{V=2^k> T^{1/8}\log^{11/8}T} R_V V^A \ll$$

$$T^{1+(A-4)/8}(\log T)^{(11A+4)/8} + T^{2+(A-12)/8}(\log T)^{(11A+4)/8} \ll$$

$$T^{1+(A-4)/8}(\log T)^{(11A+4)/8},$$

which gives

(7.42) $$\int_1^T |\zeta(1/2 + it)|^A dt \ll T^{1+(A-4)/8}(\log T)^{(11A+4)/8}, \quad 4 \le A \le 12,$$

and the same may be obtained (with an even slightly better log-factor) directly from (5.1),(7.15) by using Hölder's inequality for integrals.

For the range $12 \le A \le 178/13$ in Theorem 7.2 we use (7.14) to estimate R_V when $V \le T^{3/22}$ and (7.9) for $V > T^{3/22}$, obtaining similarly as in the previous case the estimate $M(A) \le 2 + 3(A-12)/22$. Finally the third estimate in (7.37) will follow from a more general result, viz.

(7.43) $$S = \sum_{r\le R} |\zeta(1/2 + it_r)|^A \ll T^{c(A-6)+1+\varepsilon}, \quad A \ge 178/13, \ c \ge 4/25,$$

where the t_r's satisfy (7.4) and $\zeta(1/2 + it) \ll |t|^{c+\varepsilon}$, so that by (6.63) the value $c = 35/216$ leads to $M(A) \le 1 + 35(A - 6)/216$. To see that (7.43) holds write $S = S_1 + S_2$, where in S_1 we consider the t_r's for which (7.41) holds with $V \ge T^{4/25}$, and in S_2 the t_r's for which (7.41) holds with $V < T^{4/25}$. For S_1 we have $R_V \ll T^{1+\varepsilon}V^{-6}$ by (7.36), so that summing over $O(\log T)$ values of V we obtain

$$(7.44)\qquad S_1 \ll \sum_{V=2^k\geq T^{4/25}} R_V V^A \ll T^{1+\varepsilon} \sum_{V=2^k\geq T^{4/25}} V^{(A-6)} \ll T^{c(A-6)+1+\varepsilon}.$$

In S_2 we have $R_V \ll T^{29/13+\varepsilon}V^{-178/13}$ by (7.9), and thus

$$(7.45)\qquad S_2 \ll \sum_{V=2^k<T^{4/25}} R_V V^A \ll T^{29/13+\varepsilon} \sum_{V=2^k<T^{4/25}} V^{(A-178/13)} \ll$$

$$\ll T^{(A-178/13)c+29/13+\varepsilon} \ll T^{c(A-6)+1+\varepsilon},$$

provided that $c \geq 4/25$. Combining (7.44) and (7.45) we obtain (7.43), which completes the proof of Theorem 7.2.

§3. Power moments for $1/2 < \sigma < 1$.

We suppose throughout this section that $1/2 < \sigma < 1$ is fixed, and consider power moments of the type (7.2), which will follow from discrete estimates of the type

$$(7.46)\qquad \sum_{r\leq R} |\zeta(\sigma + it_r)|^{m(\sigma)} \ll T^{1+\varepsilon}.$$

For technical reasons the following conditions will be imposed on the real numbers $t_1, t_2, \ldots, t_R$:

$$(7.47)\qquad \log^2 T \leq |t_r| \leq T \text{ for } r \leq R,\ |t_r - t_s| \geq \log^4 T \ \text{ for } r \neq s \leq R.$$

We seek an upper bound for R when

$$(7.48)\qquad |\zeta(\sigma + it_r)| \geq V > T^{\varepsilon} \ \text{ for } r \leq R,$$

and similarly as in the case $\sigma = 1/2$ an upper bound for R will lead to estimates of the type (7.46) by collecting $O(\log T)$ subsums where

$$2^k = V \leq |\zeta(\sigma + it_r)| < 2V < T^{1/6}.$$

If we choose the t_r's such that

$$|\zeta(\sigma + it_r)| = \max_{r\log^4 T\leq t\leq (r+1)\log^4 T} |\zeta(\sigma + it)|,\quad r = 1,2,\ldots$$

and then consider separately t_{2m} and t_{2m+1}, it is seen that the spacing condition required by (7.47) does hold, so that (7.46) leads to (7.2), namely

$$\int_1^T |\zeta(\sigma + it)|^{m(\sigma)} dt \ll T^{1+\varepsilon},$$

which is the desired estimate with a large number of applications, some of which will be given in Chapter 9 and Chapter 10.

Our starting point is the relation

$$(7.49) \qquad \sum_{n=1}^{\infty} d_k(n) e^{-n/Y} n^{-s} = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} Y^w \Gamma(w) \zeta^k(s+w)\, dw,$$

which is just (4.60) with $h = 1$. We shall need (7.49) with $k = 1$ or $k = 2$, and $Y = Y(r)$ will be a real number (to be suitably chosen) which satisfies $1 \ll Y \ll T^C$. For s we take $s = \sigma + it_r$, where t_r satisfies (7.47) and (7.48). Moving the line of integration in (7.49) to $\operatorname{Re} w = 1/2 - \sigma$ we encounter a pole of order k at $w = 1 - s$ with residue $O(1)$ in view of (1.32), and a simple pole at $w = 0$ with residue $\zeta^k(s)$. Therefore

$$(7.50) \qquad \sum_{n \le Y} d_k(n) e^{-n/Y} n^{-s} = \zeta^k(s) + O(1) + (2\pi i)^{-1} \int_{\operatorname{Re} w = 1/2-\sigma} \zeta^k(s+w)\Gamma(w) Y^w dw.$$

The portion of the integral in (7.50) for which $|\operatorname{Im} w| \ge \log^2 T$ is $o(1)$ by (1.32), and so for each $s = \sigma + it_r$ under consideration we have

$$(7.51) \qquad \zeta^k(\sigma + it_r) \ll 1 + \Big|\sum_{n \le Y} d_k(n) e^{-n/Y} n^{-\sigma - it_r}\Big| + \int_{-\log^2 T}^{\log^2 T} |\zeta(1/2 + it_r + iv)|^k Y^{1/2-\sigma} e^{-|v|} dv.$$

Taking into account (7.48) this implies either

$$(7.52) \qquad V^k \ll \Big|\sum_{n \le Y} d_k(n) e^{-n/Y} n^{-\sigma - it_r}\Big| \ll \log T \cdot \max_{M \le Y/2} \Big|\sum_{M < n \le 2M} d_k(n) e^{-n/Y} n^{-\sigma - it_r}\Big|$$

or

$$(7.53) \qquad V^k \ll Y^{1/2-\sigma} |\zeta(1/2 + it_r')|^k,$$

where

$$(7.54) \qquad |\zeta(1/2 + it_r')| = \max_{-\log^2 T \le v \le \log^2 T} |\zeta(1/2 + it_r + iv)|.$$

This discussion shows that the estimation of (7.2) may be reduced to a large values estimate for Dirichlet polynomials which satisfy (7.52), and a large values estimate for (7.53), which is in fact furnished by Theorem 7.1 and its corollaries. Therefore before we formulate our results concerning bounds for $m(\sigma)$, it will be useful to derive a large values estimate for Dirichlet polynomials capable of dealing with the t_r's satisfying (7.52). This estimate is contained in

Lemma 7.2. Let $t_1,\dots,t_R$ be real numbers which satisfy (7.47), $1/2 < \sigma < 1$ fixed, and let for $r \le R$

$$T^{\varepsilon} < V \le \Big|\sum_{M<n\le 2M} a(n)n^{-\sigma-it_r}\Big|, \tag{7.55}$$

where $a(n) \ll M^{\varepsilon}$, $1 \ll M \ll T^C$, $C > 0$. Then

$$R \ll T^{\varepsilon}(M^{2-2\sigma}V^{-2} + TV^{-f(\sigma)}), \tag{7.56}$$

where

$$\begin{aligned} f(\sigma) &= 2/(3-4\sigma) && \text{for } 1/2 < \sigma \le 2/3,\\ f(\sigma) &= 10/(7-8\sigma) && \text{for } 2/3 \le \sigma \le 11/14,\\ f(\sigma) &= 34/(15-16\sigma) && \text{for } 11/14 \le \sigma \le 13/15,\\ f(\sigma) &= 98/(31-32\sigma) && \text{for } 13/15 \le \sigma \le 57/62,\\ f(\sigma) &= 5/(1-\sigma) && \text{for } 57/62 \le \sigma \le 1-\varepsilon. \end{aligned} \tag{7.57}$$

Proof of Lemma 7.2. The expected bound in (7.56) is $R \ll T^{\varepsilon}M^{2-2\sigma}V^{-2}$, and $TV^{-f(\sigma)}$ is the extra term which may be thought of as an error term. We start from the inequality (1.35), taking $\xi = \{\xi_n\}_{n=1}^{\infty}$, where $\xi_n = a(n)b^{-1/2}(n)n^{-\sigma}$ for $M < n \le 2M$ and zero otherwise, and $\varphi_r = \{\varphi_{r,n}\}_{n=1}^{\infty}$ where $\varphi_{r,n} = b^{1/2}(n)n^{-it_r}$, $b(n) = e^{-(n/2M)^h} - e^{-(n/M)^h}$ and $h = \log^2 T$. Then $(\varphi_r,\varphi_s) = H(it_r - it_s)$, where by (1.39)

$$H(it) = \sum_{n=1}^{\infty} b(n)n^{-it} = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty} \zeta(w+it)\Gamma(1+\tfrac{w}{h})((2M)^w - M^w)w^{-1}\,dw, \tag{7.58}$$

which follows from (4.60) with $Y = 2M$ and $Y = M$ respectively on subtracting. Note that for $M < n \le 2M$ we have $1 \ll b(n) \ll 1$, $H(0) \ll M$, $\|\xi\|^2 \ll T^{\varepsilon}M^{1-2\sigma}$, and the integrand in (7.58) is regular for $\operatorname{Re} w > -h$, except for a simple pole at $w = 1 - it$ with residue $O(T^{-c})$ for any fixed $c > 0$ if $|t| \gg \log^3 T$. Recalling that $c(\theta)$ is the function defined by (6.51) to satisfy $\zeta(\theta + it) \ll t^{c(\theta)+\varepsilon}$, it is seen that using properties of $c(\theta)$ discussed in §5 of Chapter 6 we have

$$(7.59)\qquad \begin{aligned} c(\theta) &= 1/2 - \theta && \text{for } \theta \le 0,\\ c(\theta) &= (3 - 4\theta)/6 && \text{for } 0 \le \theta \le 1/2,\\ c(\theta) &= (7 - 8\theta)/18 && \text{for } 1/2 \le \theta \le 5/7,\\ c(\theta) &= (15 - 16\theta)/50 && \text{for } 5/7 \le \theta \le 5/6,\\ c(\theta) &= (1 - \theta)/5 && \text{for } 5/6 \le \theta \le 1, \end{aligned}$$

where we used (6.59) with $l = 4$ and $l = 5$.

To estimate $H(it)$ in (7.58) we move the line of integration to $\operatorname{Re} w = \theta$, where

$$(7.60)\qquad \begin{aligned} \theta &= (3\sigma - 2)/(2\sigma - 1) && \text{for } 1/2 < \sigma_0 \le \sigma \le 2/3,\\ \theta &= (9\sigma - 6)/(4\sigma - 1) && \text{for } 2/3 \le \sigma \le 11/14,\\ \theta &= (25\sigma - 16)/(8\sigma + 1) && \text{for } 11/14 \le \sigma \le 13/15,\\ \theta &= (65\sigma - 40)/(16\sigma + 9) && \text{for } 13/15 \le \sigma \le 57/62,\\ \theta &= (12\sigma - 7)/(2\sigma + 3) && \text{for } 57/62 \le \sigma \le 1 - \varepsilon, \end{aligned}$$

so that the values of θ lie in the ranges $\theta \le 0$, $0 \le \theta \le 1/2$, $1/2 \le \theta \le 5/7$, $5/7 \le \theta \le 5/6$, $5/6 \le \theta \le 1$ respectively and so (7.59) may be used. Using (1.32) we obtain for $r \ne s$

$$(7.61)\qquad H(it_r - it_s) \ll T^{\varepsilon}\int_{-L^2}^{L^2} |\zeta(\theta+iv+it_r-it_s)|e^{-|v|/h}M^{\theta}dv + o(1) \ll T^{c(\theta)+\varepsilon}M^{\theta} + o(1).$$

Therefore (1.35) gives

$$R^2V^2 \ll T^{\varepsilon}M^{1-2\sigma}(RM + \sum_{r\ne s}|H(it_r - it_s)|),$$

and (7.60) leads to

$$(7.62)\qquad R \ll T^{\varepsilon}(M^{2-2\sigma}V^{-2} + RM^{\theta+1-2\sigma}T^{c(\theta)}V^{-2}) \ll T^{\varepsilon}M^{2-2\sigma}V^{-2},$$

provided that

$$(7.63)\qquad T = T_0 = V^{(2-\varepsilon)/c(\theta)}M^{(2\sigma-1-\theta)/c(\theta)},$$

since $V > T^{\varepsilon}$ by hypothesis. If (7.62) is not satisfied it may be observed that if in (7.55) t_r is replaced by $t_r + T_0$ for any fixed T_0, then $a(n)$ is replaced by $a_0(n) = a(n)n^{-iT_0}$, and $|a_0(n)| = |a(n)| \ll M^{\varepsilon}$. Hence if the t_r's lie in an interval of length not exceeding T_0, then $R \ll T^{\varepsilon}M^{2-2\sigma}V^{-2}$, and dividing T into subintervals of length at most T_0 (where T_0 is given by (7.63)) we obtain

$$(7.64)\qquad R \ll T^{\varepsilon}M^{2-2\sigma}V^{-2}(1 + T/T_0) \ll$$

$$T^{\varepsilon}(M^{2-2\sigma}V^{-2} + TM^{(2c(\theta)+1+\theta-2\sigma(1+c(\theta))/c(\theta)}V^{-2(1+c(\theta))/c(\theta)}).$$

With $c(\theta)$ and θ given by (7.59) and (7.60) it is readily checked that

$$2c(\theta) + 1 + \theta - 2(1 + c(\theta))\sigma = 0, \quad 2(1 + c(\theta))/c(\theta) = f(\sigma),$$

where $f(\sigma)$ is given by (7.57), and thus (7.56) follows.

Having Theorem 7.1 and Lemma 7.2 at our disposal we are ready now to state and prove the main result of this section, which is

THEOREM 7.3. Let $m(\sigma)$ be defined for each fixed $1/2 < \sigma < 1$ by (7.2). Then

$$
\begin{aligned}
&m(\sigma) \geq 4/(3 - 4\sigma) && \text{for } 1/2 < \sigma \leq 5/8,\\
&m(\sigma) \geq (48\sigma - 6)/(7 - 8\sigma)(4\sigma - 1) && \text{for } 5/8 \leq \sigma \leq 5/7,\\
&m(\sigma) \geq (208\sigma - 70)/(15 - 16\sigma)(4\sigma - 1) && \text{for } 5/7 \leq \sigma \leq 5/6, \qquad (7.65)\\
&m(\sigma) \geq (28\sigma - 13)/(4\sigma - 1)(1 - \sigma) && \text{for } 5/6 \leq \sigma \leq 13/15,\\
&m(\sigma) \geq 98/(31 - 32\sigma) && \text{for } 13/15 \leq \sigma \leq 0.91591\ldots,\\
&m(\sigma) \geq (24\sigma - 9)/(4\sigma - 1)(1 - \sigma) && \text{for } 0.91591\ldots \leq \sigma \leq 1 - \varepsilon.
\end{aligned}
$$

In addition we have $m(35/54) \geq 9$, $m(41/60) \geq 10$, $m(7/10) \geq 11$, $m(5/7) \geq 12$, $m(2/3) \geq 9.6187\ldots$, $m(3/4) \geq 528/37 = 14.270270\ldots$, $m(5/6) \geq 188/7$, $m(7/8) \geq 36.8$.

Proof of Theorem 7.3. We begin the proof by considering first the range $1/2 < \sigma \leq 5/8$ and proving $m(\sigma) \geq 4/(3 - 4\sigma)$ (this holds also for $\sigma = 1/2$ by (5.1)). In view of the discussion made at the beginning of this section it will be sufficient to prove

$$R \ll T^{1+\varepsilon}V^{-4/(3-4\sigma)} \qquad (7.66)$$

for the number of points t_r which satisfy (7.47) and (7.48). To simplify writing we shall omit in the rest of this proof factors like $T^{\varepsilon}\log^{C}T$ on right-hand sides of inequalities implied by $\ll$. To obtain (7.66) we consider separately subsets A and B of $\{t_r\}$ such that $t_r \in A$ if V in (7.48) satisfies $V \leq T^{(3-4\sigma)/8}$ and $t_r \in B$ if $V > T^{(3-4\sigma)/8}$. If $R_1 = |A|$, $R_2 = |B|$, then $R = R_1 + R_2$ and

$$R_1 \ll Y_1^{2-2\sigma}V^{-4} + TV^{-4/(3-4\sigma)} + Y_1^{1/2-\sigma}V^{-2}\sum_{t_r\in A}|\zeta(1/2 + it_r')|^2, \qquad (7.67)$$

which follows from (7.52) and (7.53) with $k = 2$ when one applies Lemma 7.2. Here $M \leq Y/2 = Y_1/2$ is chosen in such a way that $\gg 1/\log T$ numbers $t_r \in A$ satisfy (7.52) with that particular M. Using $M(4) = 1$ and the Cauchy-Schwarz inequality we have

$$(7.68)\qquad R_1 \ll Y_1^{2-2\delta}V^{-4} + TV^{-4/(3-4\delta)} + TV^{-4}Y_1^{1-2\delta},$$

and in view of $V \le T^{(3-4\delta)/8}$ the choice $Y_1 = T$ gives

$$(7.69)\qquad R_1 \ll TV^{-4/(3-4\delta)} + T^{2-2\delta}V^{-4} \ll TV^{-4/(3-4\delta)}.$$

To bound R_2 we reason analogously, only now we use $M(12) \le 2$ and Hölder's inequality to obtain

$$R_2 \ll Y_2^{2-2\delta}V^{-4} + TV^{-4/(3-4\delta)} + Y_2^{1/2-\delta}R_2^{5/6}V^{-2}\Big(\sum_{t_r'\in B} |\zeta(1/2 + it_r')|^{12}\Big)^{1/6},$$

$$(7.70)\qquad R_2 \ll Y_2^{2-2\delta}V^{-4} + TV^{-4/(3-4\delta)} + Y_2^{3-6\delta}T^2V^{-12}.$$

Choosing $Y_2 = T^{2/(4\delta-1)}V^{-8/(4\delta-1)} \gg 1$ we have

$$(7.71)\qquad R_2 \ll TV^{-4/(3-4\delta)} + T^{(4-4\delta)/(4\delta-1)}V^{-12/(4\delta-1)}.$$

The second term on the right-hand side of (7.71) does not exceed the first if

$$T^{(5-8\delta)/(4\delta-1)} \le V^{8(5-8\delta)/(4\delta-1)(3-4\delta)},$$

and this condition is satisfied since $1/2 < \delta \le 5/8$ and $V > T^{(3-4\delta)/8}$. Thus from (7.69) and (7.71) we obtain

$$(7.72)\qquad R = R_1 + R_2 \ll TV^{-4/(3-4\delta)},$$

implying $m(\delta) \ge 4/(3 - 4\delta)$ for $1/2 < \delta \le 5/8$ as asserted.

We consider now the range $5/8 \le \delta \le 2/3$, and let this time A and B denote subsets of $\{t_r'\}$ (see (7.53) and (7.54)) such that in (7.8)

$$(7.73)\qquad R \ll TV^{-6}$$

and

$$(7.74)\qquad R \ll T^{(p+q)/p}V^{-2(1+2p+2q)/p}$$

hold respectively for $R = R_1 = |A|$ and $R = R_2 = |B|$. In applying (7.73) and (7.74) we have to replace V by $VY^{(2\delta-1)/4}$ in view of (7.53) with $k = 2$. Therefore we have

$$R_1 \ll Y_1^{2-2\delta}V^{-4} + TV^{-4/(3-4\delta)} + Y_1^{(3-6\delta)}TV^{-6},$$

where Lemma 7.2 was used again. With the choice $Y_1 = (TV^{-2})^{2/(1+2\delta)} \gg 1$ the above estimate becomes

(7.75) $$R_1 \ll TV^{-4/(3-4\sigma)} + T^{4(1-\sigma)/(1+2\sigma)}V^{-12/(1+2\sigma)}$$

Analogously using (7.74) it follows that

(7.76) $$R_2 \ll Y_2^{2-2\sigma}V^{-4} + TV^{-4/(3-4\sigma)} + T^{(p+q)/p}V^{-2(1+2p+2q)/p}Y_2^{(1/2-\sigma)(1+2p+2q)/p},$$

and we shall choose Y_2 to satisfy

$$Y_2 = T^{2(p+q)/((2+4q)\sigma-1+2p-2q)}V^{-4(1+2q)/((2+4q)\sigma-1+2p-2q)},$$

so that the first and the third term on the right-hand side of (7.76) are equal. Since by (7.59) we have $c(\theta) \le 1/8$ for $\theta \ge 5/8$ and $2(p+q)/4(1+2q) \ge 1/8$ the condition $Y_2 \gg 1$ will be satisfied, and hence from (7.75) and (7.76)

(7.77) $$R \ll TV^{-4/(3-4\sigma)} + T^{\frac{4-4\sigma}{1+2\sigma}}V^{\frac{-12}{1+2\sigma}} + T^{\frac{4(1-\sigma)(p+q)}{(2+4q)\sigma-1+2p-2q}}V^{\frac{-4(1+2p+2q)}{(2+4q)\sigma-1+2p-2q}}.$$

The exponent of T of the last term above equals unity for

(7.78) $$\sigma = (1+2p+6q)/(2+4p+8q),$$

giving

(7.79) $$R \ll TV^{-4/(3-4\sigma)} + T^{(4-4\sigma)/(1+2\sigma)}V^{-12/(1+2\sigma)} + TV^{-F},$$

$$F = 2(1+2p+2q)(1+2p+4q)/(p+q+4pq+2p^2+2q^2).$$

In (7.79) the term TV^{-F} is the largest, which will be shown now for $\sigma = 2/3$. In that case (7.78) reduces to $q - p = 1/2$, and thus with the exponent pair $(p,q) = (\alpha/2+\varepsilon, \alpha/2+1/2+\varepsilon)$, $\alpha = 0.3290213568\ldots$ one obtains

(7.80) $$R \ll TV^{-9.61872\ldots} + (TV^{-9})^{4/7},$$

and one has $(TV^{-9})^{4/7} \le TV^{-x}$ for

(7.81) $$V \le T^{3/(7x-36)}.$$

Since by (7.59) one has $c(2/3) \le 5/54$, it is seen that (7.81) is certainly satisfied for $5/54 \le 3/(7x-36)$, or $x \le 342/35 = 9.7714\ldots$. This proves $m(2/3) \ge 9.61872\ldots$, which is actually the optimal bound this method allows. With $(p,q) = (2/7, 4/7)$ in (7.78) we obtain $\sigma = 35/54 = 0.6481481\ldots$, and a calculation similar to the one above gives $m(35/54) \ge 9$. The above procedure may be also used when $\sigma \ge 2/3$, only in view of Lemma 7.2 the first term in (7.77) is to be replaced by $TV^{-2f(\sigma)}$, i.e. we have

(7.82) $$R \ll TV^{-2f(\sigma)} + T^{\frac{4-4\sigma}{1+2\sigma}}V^{\frac{-12}{1+2\sigma}} + T^{\frac{(4-4\sigma)(p+q)}{(2+4q)\sigma-1+2p-2q}}V^{\frac{-4(1+2p+2q)}{(2+4q)\sigma-1+2p-2q}}.$$

Calculations for $\sigma \geq 2/3$ are carried out in the manner described above; the term $TV^{-2f(\sigma)}$ is always the smallest one, and the second and third term in (7.82) do not exceed TV^{-x} and TV^{-y} respectively for values of x and y which will depend on $c(\theta)$, where for $c(\theta)$ we use the bounds furnished by (7.59). With $(p,q) = B(2/7,4/7) = (1/14,11/14)$ (here B denotes the operator defined by Lemma 2.9) the last term in (7.82) is TV^{-10} for $\sigma = 41/60 = 0.68333...$, and since then the other two terms in (7.82) are smaller, we obtain $m(41/60) \geq 10$. Using $(p,q) = (2/7,4/7)$ and $\sigma = 7/10$, $\sigma = 5/7$ we obtain likewise $m(7/10) > 11$ and $m(5/7) > 12$ respectively. For $\sigma \geq 3/4$ we have from (7.82) that the first and the third term are $\ll TV^{-x}$ for

$$x \leq 8(3 + 6p + 2q)/(1 + 4p + 2q), \tag{7.83}$$

where we used $c(3/4) \leq 1/16$. The choice $(p,q) = (5/24,15/24)$ gives $x \leq 528/37 = 14.270270...$, so that $m(3/4) \geq 528/37$, since the middle term in (7.82) turns out to be $T^{2/5}V^{-24/5} < TV^{-y}$ for $y \leq 72/5 = 14.4$. Similarly one obtains $m(5/6) \geq 188/7 = 26.857142...$ for $(p,q) = (2/7,4/7)$ and likewise $m(7/8) \geq 36.8$.

To finish the proof of Theorem 7.3 it remains to prove the general estimate for $m(\sigma)$ when $\sigma > 5/8$, as given by (7.65). For $5/8 \leq \sigma \leq 13/15$ we use Lemma 7.2 and $M(12) \leq 2$ to obtain as before

$$R \ll TV^{-2f(\sigma)} + Y^{2-2\sigma}V^{-4} + Y^{3-6\sigma}T^{2}V^{-12} \ll \tag{7.84}$$

$$\ll TV^{-2f(\sigma)} + T^{(4-4\sigma)/(4\sigma-1)}V^{-12/(4\sigma-1)}$$

for $Y = T^{2/(4\sigma-1)}V^{-8/(4\sigma-1)}$. Using estimates for $c(\theta)$ furnished by (7.59) when $5/8 \leq \theta \leq 13/15$ it is seen that the last term in (7.84) is $\ll TV^{-x}$ for $x = m(\sigma)$ given by (7.65), while $TV^{-2f(\sigma)} \leq TV^{-m(\sigma)}$.

To obtain general estimates for $m(\sigma)$ when $\sigma \geq 13/15$ we shall use (7.51) with $k = 1$, since for that range the values of $f(\sigma)$ given by Lemma 7.2 are large enough for our purposes and the estimate $TV^{-f(\sigma)}$ suffices, whereas for smaller values of σ it was necessary to use $k = 2$ in (7.51), with the effect that V in Lemma 7.2 is replaced by V^2. To avoid tedious calculations we choose $(p,q) = (\frac{2}{7},\frac{4}{7})$ in (7.74) and let similarly as before A and B denote subsets of $\{t_r'\}$ for which $R \ll TV^{-6}$ and $R \ll T^3V^{-19}$ respectively hold with $R_1 = |A|$ and $R_2 = |B|$. Applying then Lemma 7.2 we obtain

$$R_1 \ll Y_1^{2-2\sigma}V^{-2} + Y_1^{3-6\sigma}TV^{-6} + TV^{-f(\sigma)},$$

$$R_2 \ll Y_2^{2-2\sigma}V^{-2} + Y_2^{19(1/2-\sigma)}T^3V^{-19} + TV^{-f(\sigma)}.$$

If we choose $Y_1 = (TV^{-4})^{1/(4\sigma-1)} \gg 1$ and $Y_2 = (T^6V^{-34})^{1/(34\sigma-15)} \gg 1$, then

$$R_1 \ll T^{(2-2\sigma)/(4\sigma-1)}V^{-6/(4\sigma-1)} + TV^{-f(\sigma)}, \tag{7.85}$$

$$R_2 \ll T^{(12-12\sigma)/(34\sigma-15)}V^{-38/(34\sigma-15)} + TV^{-f(\sigma)}. \tag{7.86}$$

With $c(\theta) \le (1-\theta)/5$ for $\theta \ge 5/6$ we obtain $R_1 + R_2 \ll TV^{-x}$ for

$$x = \min\left(f(\sigma), \frac{24\sigma - 9}{(4\sigma - 1)(1 - \sigma)}, \frac{192\sigma - 97}{(34\sigma - 15)(1 - \sigma)}\right), \tag{7.87}$$

where $f(\sigma) = 98/(31 - 32\sigma)$ for $13/15 \le \sigma \le 57/62 = 0.91935..$ and $f(\sigma) = 5/(1-\sigma)$ for $57/62 \le \sigma \le 1 - \varepsilon$. Now for $13/15 \le \sigma \le 1$ we have $(24\sigma - 9)/(4\sigma - 1) \le 5$ and also the second term in (7.87) does not exceed the third. For $57/62 \ge \sigma \ge 0.91591...$, we have $(24\sigma - 9)/(4\sigma - 1)(1-\sigma) \le 98/(31 - 32\sigma) = f(\sigma)$, hence the last part of the theorem. In particular we have

$$m(\sigma) \ge 4.873/(1-\sigma), \quad 0.91591... \le \sigma \le 1 - \varepsilon. \tag{7.88}$$

§4. Asymptotic formulas for power moments when $1/2 < \sigma < 1$

We conclude this chapter by considering the asymptotic formula

$$\int_1^T |\zeta(\sigma + it)|^{2k}dt = T\sum_{n=1}^{\infty} d_k^2(n)n^{-2\sigma} + R(k,\sigma;T), \tag{7.89}$$

where $k \ge 1$ is a fixed integer, $1/2 < \sigma < 1$ is fixed and $R(k,\sigma;T)$ is supposed to be an error term, i.e. $R(k,\sigma;T) = o(T)$ as $T \to \infty$. This may be compared with our approach in §3, where only upper bounds of the form (7.2) were investigated and Theorem 7.3. was derived. However results of the type (7.89) may be obtained exactly with the use of Theorem 7.3, and we begin by proving

Lemma 7.3. For $k \geq 1$ a fixed integer let $1/2 \leq \sigma_k^* < 1$ denote such a number for which

$$\int_1^T |\zeta(\sigma_k^* + it)|^{2k} dt \ll T^{1+\varepsilon}$$

holds. Then the asymptotic formula (7.89) holds for $\sigma > \sigma_k^*$.

Proof of Lemma 7.3. We shall prove the lemma and obtain an explicit O-estimate for $R(k,\sigma;T)$. We have with $s = \sigma + it$

$$(7.90)\quad \int_1^T |\zeta(s)|^{2k} dt = \int_1^T \Big|\sum_{n\leq T} d_k(n)n^{-s}\Big|^2 dt + O\Big(\int_1^T \Big|\zeta^{2k}(s) - \Big(\sum_{n\leq T} d_k(n)n^{-s}\Big)^2\Big| dt\Big),$$

and using Theorem 5.2 it follows that

$$(7.91)\quad \int_1^T \Big|\sum_{n\leq T} d_k(n)n^{-s}\Big|^2 dt = T\sum_{n\leq T} d_k^2(n)n^{-2\sigma} + O\Big(\sum_{n\leq T} d_k^2(n)n^{1-2\sigma}\Big) =$$

$$T\sum_{n=1}^{\infty} d_k^2(n)n^{-2\sigma} + O(T^{2-2\sigma+\varepsilon}),$$

so that the main contribution in (7.89) comes from the first term on the right-hand side of (7.90). Let now

$$F(s) = \zeta^{2k}(s) - \Big(\sum_{n\leq T} d_k(n)n^{-s}\Big)^2.$$

Since k is an integer, $F(s)$ is regular for $\mathrm{Re}\, s \geq 1/2$, $\mathrm{Im}\, s \geq 1$ and thus for $1/2 \leq \alpha < \sigma < \beta$ we may use the well-known convexity estimate

$$(7.92)\qquad \int_1^T |F(\sigma + it)| dt \ll \Big(\int_1^T |F(\alpha + it)| dt\Big)^{\frac{\beta-\sigma}{\beta-\alpha}} \Big(\int_1^T |F(\beta + it)| dt\Big)^{\frac{\sigma-\alpha}{\beta-\alpha}}.$$

We shall take $\alpha = \sigma_k^* + \delta$, $\beta = 1 + \delta$, where $0 < \delta < 1/2$ is a fixed constant which may be chosen arbitrarily small. Since k is fixed we have then

$$(7.93)\qquad \frac{\beta - \sigma}{\beta - \alpha} = \frac{1 + \delta - \sigma}{1 - \sigma_k^*} \leq \frac{1 - \sigma}{1 - \sigma_k^*} + \delta^{1/2},$$

$$(7.94)\qquad \frac{\sigma - \alpha}{\beta - \alpha} = \frac{\sigma - \sigma_k^* - \delta}{1 - \sigma_k^*} \leq \frac{\sigma - \sigma_k^*}{1 - \sigma_k^*}.$$

Using Theorem 5.2 and the definition of σ_k^* we have

$$\int_1^T |F(\alpha + it)| dt \leq \int_1^T |\zeta(\sigma_k^* + \delta + it)|^{2k} dt + \int_1^T \Big|\sum_{n\leq T} d_k(n)n^{-\sigma_k^* - \delta - it}\Big|^2 dt \ll$$

$$\ll T^{1+\delta} + T^{2-2\sigma_k^*+\varepsilon} \ll T^{1+\delta}.$$

To estimate the second integral on the right-hand side of (7.92) recall that

$$\sum_{ab=n} d_k(a)d_k(b) = d_{2k}(n),$$

hence

$$F(\sigma + it) = \sum_{n=1}^{\infty} d_{2k}(n)n^{-1-\delta-it} - (\sum_{n\le T} d_k(n)n^{-1-\delta-it})^2 = \sum_{n>T} g_k(n)n^{-1-\delta-it},$$

where $|g_k(n)| \le d_{2k}(n) \ll n^{\varepsilon}$. Therefore with Theorem 5.2 again and the Cauchy-Schwarz inequality it is seen that

$$\int_1^T |F(\sigma + it)|\,dt \le T^{1/2}(\int_1^T |\sum_{n>T} g_k(n)n^{-1-\delta-it}|^2 dt)^{1/2} \ll T^{1/2}.$$

Taking into account (7.93) and (7.94) we obtain from (7.92)

(7.95) $$\int_1^T |F(\sigma + it)|\,dt \ll T^{A},$$

(7.96) $$A = (1 + \delta)(\frac{1-\sigma}{1-\sigma_k^*} + \delta^{1/2}) + \frac{\sigma - \sigma_k^*}{2 - 2\sigma_k^*} \le \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} + \varepsilon$$

for any $\varepsilon > 0$ if $\delta = \delta(\varepsilon)$ is sufficiently small. As

$$2 - 2\sigma < \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} < 1,$$

this means that we have proved

(7.97) $$\int_1^T |\zeta(\sigma + it)|^{2k} dt = T\sum_{n=1}^{\infty} d_k^2(n)n^{-2\sigma} + O(T^{\frac{2-\sigma-\sigma_k^*}{2-2\sigma_k^*}+\varepsilon}),$$

so that (7.89) holds with

$$R(k,\sigma;T) \ll T^{(2-\sigma-\sigma_k^*)/(2-2\sigma_k^*)+\varepsilon} = o(T)$$

for $\sigma_k^* < \sigma < 1$ fixed.

Recalling that by the definition of σ_k^* and Theorem 7.3 we may take $\sigma_2^* = 1/2$, $\sigma_3^* = 7/12$, $\sigma_4^* = 5/8$, $\sigma_5^* = 41/60$, $\sigma_6^* = 5/7$, we obtain special estimates from (7.97), which we formulate as

THEOREM 7.4.

$$\int_1^T |\zeta(\sigma + it)|^4 dt = T\sum_{n=1}^{\infty} d_2^2(n)n^{-2\sigma} + O(T^{3/2-\sigma+\varepsilon}), \quad 1/2 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma + it)|^6 dt = T\sum_{n=1}^{\infty} d_3^2(n)n^{-2\sigma} + O(T^{(17-12\sigma)/10+\varepsilon}), \quad 7/12 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma + it)|^8 dt = T\sum_{n=1}^{\infty} d_4^2(n)n^{-2\sigma} + O(T^{(11-8\sigma)/6+\varepsilon}),\ 5/8 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma + it)|^{10} dt = T\sum_{n=1}^{\infty} d_5^2(n)n^{-2\sigma} + O(T^{(79-60\sigma)/38+\varepsilon}),\ 41/60 < \sigma < 1,$$

$$\int_1^T |\zeta(\sigma + it)|^{12} dt = T\sum_{n=1}^{\infty} d_6^2(n)n^{-2\sigma} + O(T^{(9-7\sigma)/4+\varepsilon}),\ 5/7 < \sigma < 1.$$

Naturally explicit bounds for $R(k,\sigma;T)$ in (7.89) when $k > 6$ may be also obtained by this method, but a general formula would be rather complicated in view of Theorem 7.3, and therefore it seems reasonable to consider (7.89) explicitly for small values of k only.

Finally it may be mentioned that one can also investigate power moments of $|\zeta(\sigma+it)|^m$ when $m = 1$ or $m = 2$ by the above method and obtain

(7.98) $$\int_1^T |\zeta(\sigma + it)| dt = T\sum_{n=1}^{\infty} d_{1/2}^2(n)n^{-2\sigma} + O(T^{5/4-\sigma/2+\varepsilon}),\ 1/2 < \sigma < 1,$$

(7.99) $$\int_1^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma)T + O(T^{2-2\sigma}),\ 1/2 < \sigma < 1,$$

where $d_{1/2}(n)$ is the special case of the generalized divisor function $d_z(n)$, which is defined by

$$\zeta^z(s) = \sum_{n=1}^{\infty} d_z(n)n^{-s}, \quad \mathrm{Re}\, s > 1.$$

For the proof of (7.98) and (7.99) we shall use the simplest form of the approximate functional equation, namely

(7.100) $$\zeta(s) = \sum_{n\le x} n^{-s} + x^{1-s}/(s-1) + O(x^{-\sigma}),\ x > t/\pi > t_0,\ 0 < \sigma < 1.$$

To obtain (7.98) and (7.99) it will be sufficient to obtain the corresponding formula for the interval $[T/2,T]$. We take in (7.100) $x = T$, obtaining

(7.101) $$\zeta(s) = \sum_{n\le T} n^{-s} + O(T^{-\sigma}),$$

where $s = \sigma + it$, $T/2 \le t \le T, 1/2 < \sigma < 1$. To obtain (7.98) we write

$$\sum_{n\le T} n^{-s} = \Big(\sum_{n\le T^{1/2}} d_{1/2}(n)n^{-s}\Big)^2 + \sum_{T^{1/2}<n\le T} g(n)n^{-s},$$

where clearly $|g(n)| \le 1$. Then we have

$$(7.102)\qquad \int_{T/2}^{T} |\zeta(s)|dt = \int_{T/2}^{T} |\sum_{n\le T^{1/2}} d_{1/2}(n)n^{-s}|^2dt + O(\int_{T/2}^{T} |\sum_{T^{1/2}<n\le T} g(n)n^{-s}|dt) + O(T^{1-\sigma}).$$

Using the Cauchy-Schwarz inequality and the same argument as in the proof of Lemma 7.3 (only with $\alpha = 1/2 + \delta$) we obtain that the first O-term on the right-hand side of (7.102) is $\ll T^{5/4-\sigma-2+\varepsilon}$ and (7.98) follows on applying Theorem 5.2 to the first term on the right-hand side of (7.102).

The proof of (7.99) is even easier. Namely from (7.101) we have at once

$$\int_{T/2}^{T} |\zeta(s)|^2dt = \int_{T/2}^{T} |\sum_{n\le T} n^{-s}|^2dt + O(T^{-\sigma}\int_{T/2}^{T} |\sum_{n\le T} n^{-s}\, dt|) + O(T^{1-2\sigma}).$$

Theorem 5.2 and the Cauchy-Schwarz inequality give

$$\int_{T/2}^{T} |\sum_{n\le T} n^{-s}|^2dt = \frac{1}{2}T\sum_{n\le T} n^{-2\sigma} + O(\sum_{n\le T} n^{1-2\sigma}) = \frac{1}{2}\zeta(2\sigma)T + O(T^{2-2\sigma}),$$

$$\int_{T/2}^{T} |\sum_{n\le T} n^{-s}|dt \le T^{1/2}(\int_{T/2}^{T} |\sum_{n\le T} n^{-s}|^2dt)^{1/2} \ll T,$$

so that we obtain

$$\int_{T/2}^{T} |\zeta(\sigma + it)|^2dt = \frac{1}{2}\zeta(2\sigma)T + O(T^{2-2\sigma}),$$

hence (7.99).

N O T E S

The main results of this chapter involve upper bounds for power moments of the form (7.1) or (7.2). However when one asks for lower bounds in the same problem the situation is different. Already in Chapter 7 of Titchmarsh [8] one can find the bound ($k \ge 1$ is a fixed integer)

$$\int_{0}^{T} |\zeta(\tfrac{1}{2} + it)|^{2k}e^{-t/T}dt \gg T(\log T)^{k^2},$$

whence

$$\int_{0}^{T} |\zeta(\tfrac{1}{2} + it)|^{2k}dt = \Omega(T(\log T)^{k^2}).$$

A substantial advance has been made recently by K. Ramachandra [4],[5], who proved

$$(7.103)\qquad \int_{T-H}^{T+H} |\zeta(\tfrac{1}{2} + it)|^{k}dt > C_kH(\log H)^{k^2/4}$$

for $k \geq 1$ a fixed integer, $C_k > 0$ a constant depending on k only, $T \geq T_0$, $H > \log^{\epsilon} T$. In fact Ramachandra deduced (7.103) from a general estimate which also gives good bounds when ζ is replaced by $\zeta^{(m)}$ in (7.103). The bound (7.103) with k = 1 will be used in the next chapter, where a simple proof of a weaker result than (7.103) will be given. Ramachandra has also proved

$$\int_{T-H}^{T+H} |\zeta(1/2 + it)|dt \ll H\log^{1/4}H, \quad T^{\epsilon} \leq H \leq T,$$

and on the other hand it may seem plausible to conjecture

$$\int_0^T |\zeta(1/2 + it)|^{2k}dt = (D_k + o(1))T(\log T)^{k^2}$$

for $k \geq 3$ and some $D_k > 0$. However it is hard even to give a heuristic value of the constant D_k.

Following an approach that uses convexity arguments of R.M. Gabriel [1], D.R. Heath-Brown [7] succeeded to obtain results in a manner that is simpler than Ramachandra's [4], [5] , and in some cases his results are also sharper. Thus for H = T Heath-Brown proved unconditionally that (7.103) holds for all rational $k > 0$ and for all real $k > 0$ if the Riemann hypothesis is true.

This discussion shows that the gap between the best upper and lower bounds for power moments higher than the fourth is still very large, and the (expected) result M(6) = 1 would be a big advance in zeta-function theory.

An alternative approach to (7.6) is via the approximate functional equation (4.11), where we let $T/2 \leq t \leq T$, $s = 1/2 + it$, $T/4\pi \leq x \leq T/2\pi$. We have then

$$|\zeta(1/2+it)|^4 \ll \Big|\sum_{n\leq x} d(n)n^{-1/2-it}\Big|^2 + \Big|\sum_{n\leq t^2/4\pi^2 x} d(n)n^{-1/2-it}\Big|^2 + \log^2 T,$$

and multiplying by dx/x and integrating from $T/4\pi$ to $T/2\pi$ we obtain

$$|\zeta(1/2+it)|^4 \ll \int_{T/4\pi}^{T/2\pi} \Big|\sum_{n\leq x} d(n)n^{-1/2-it}\Big|^2 \frac{dx}{x} + \int_{T/4\pi}^{T/2\pi} \Big|\sum_{n\leq t^2/4\pi^2 x} d(n)n^{-1/2-it}\Big|^2 \frac{dx}{x} + \log^2 T,$$

and the change of variable $y = t^2/4\pi^2 x$ in the second integral above leads to

$$|\zeta(1/2 + it)|^4 \ll \int_{T/8\pi}^{T/\pi} \Big|\sum_{n\leq x} d(n)n^{-1/2-it}\Big|^2 \frac{dx}{x} + \log^2 T.$$

In this expression the length of the sum does not depend on t, so that summing over $t = t_r$ and using the mean value theorem (5.14) we obtain (7.6) as before. However for $k \geq 6$ the lengths of the Dirichlet polynomials are too large for (any known form of) the approximate functional equation to produce a good upper bound for $\int_0^T |\zeta(1/2+it)|^k dt$, and here we use a new method, based on the mean value estimate for integrals over short intervals (Theorem 6.2), which was introduced by D.R. Heath-Brown [1].

All the results of §2 and §3 are to be found in the author's paper [2], which generalizes and sharpens the results of Heath-Brown's paper [1] where the first proof of the important result (7.15) is given (as mentioned in Chapter 6, a different proof has been given recently by H. Iwaniec [2]),but where power moments for $\sigma > 1/2$ are not treated. The use of the theory of exponent pairs, as embodied in the estimate of Lemma 7.1, makes it possible to obtain satisfactory results both for $\sigma = 1/2$ and $1/2 < \sigma < 1$. In the latter case a novel feature is Lemma 7.2, which enables one to deal effectively with large values of the Dirichlet polynomials appearing in (7.52). Bounds given here for power moments are the sharpest ones hitherto, and improve much on estimates of Chapter 7 of Titchmarsh [8].

It may be remarked that the range $V \geq T^{11/72}\log^{5/4}T$ for which the bound $R \ll TV^{-6}\log^8 T$ holds in (7.36) is very close to the optimal result the method gives. Namely (7.8) yields $R \ll TV^{-6}\log^8 T$ for $V \geq T^c$ and any $c > q/(2 - 2p + 4q)$. As mentioned several times in this text, a general method for finding the minimal value of f(p,q) (where (p,q) is an exponent pair and f is a "nice" function, say rational) has not been found yet. Thus each problem has to be tackled separately, and H.-E. Richert has kindly informed me that he has calculated

$$\min_{(p,q)} q/(2 - 2p + 4q) = 0.15274776\ldots,$$

whereas $11/72 = 0.1527\dot{7}$, so that this value is very close to the optimal one.

The useful technique of dividing T into subintervals of length $\leq T_0$ and multiplying by $1 + T/T_0$, used in the proof of Lemma 7.2, seems to have been introduced by M.N. Huxley [1], p. 117, and will be repeatedly used in Chapter 9.

Concerning Theorem 7.3 it should be remarked that the bounds given for specific values of σ in the range $5/8 < \sigma \leq 7/8$ are much better than the ones that follow from the general (7.65) (and even one has $m(\sigma) \geq (28\sigma - 13)/(4\sigma - 1)(1 - \sigma)$ for $5/6 \leq \sigma \leq 1 - \varepsilon$, which is superseded by $m(\sigma) \geq 98/(31-32\sigma)$ for $\sigma \geq 0.91143... > 13/15$). The results are the sharpest ones yet, but the estimate $m(5/8) \geq 8$ (follows from the first bound in (7.65)) has been obtained first by D.R. Heath-Brown [8] by a somewhat different approach. No effort has been made (except when $\sigma = 2/3$) to obtain the best possible estimates for $m(\sigma)$ that this method allows, as this would involve tedious computations with exponent pairs, and the possible improvements would be rather small. Also it may be mentioned that one could replace T^{ε} in (7.2) by $\log^C T$, $C = C(\sigma) \geq 0$ using the same analysis more carefully. It is only when $\sigma \to 1$ that the bounds of Theorem 7.3 are superseded by the estimate $m(\sigma) \gg (1 - \sigma)^{-3/2}$, which follows from I.M. Vinogradov's method (see H.-E. Richert [4] for the estimation of the zeta-function near the line $\sigma = 1$).

For an application of Theorem 7.3 to the asymptotic formula for powerful numbers (i.e. the numbers n with the property if a prime p divides n, then p^k for a fixed $k \geq 2$ also divides n) the reader may consult the author's paper [3]. Various other divisor problems which involve powers of the zeta-function in the corresponding generating Dirichlet series admit an approach via Theorem 7.3 too.

The results of §4 are new and hitherto unpublished, improving Satz 2. of R. Wiebelitz [1]. The idea to use a convexity argument in Lemma 7.3 may be also found in R.T. Turganaliev [1], who used it in a somewhat different context. Turganaliev namely investigated the asymptotic formula

$$(7.104) \qquad \int_0^T |\zeta(\sigma + it)|^{2\lambda} dt = T\sum_{n=1}^{\infty} d^2(n) n^{-2\sigma} + O(T^{1-\varkappa+\varepsilon}),$$

where $\varepsilon > 0$ is arbitrary, $0 < \lambda < 2$, $s = \sigma + it$, $1/2 < \sigma < 1$. He proved that (7.104) holds with some $\varkappa = \varkappa(\sigma,\lambda) > 0$, where it is understood that λ does not have to be an integer. Under the simplifying assumption that the Riemann hypothesis holds it is seen that the function

$$F(s) = \zeta^{2\lambda}(s) - (\sum_{n \leq T} d_{\lambda}(n) n^{-s})^2$$

has an analytic continuation for $\mathrm{Re}\, s > 1/2$, so that the convexity argument used in the proof of Lemma 7.3 may be applied with $\alpha = 1/2 + \delta$, giving (7.104) with $\varkappa = \sigma - 1/2$ (it has been stated without proof by Heath-Brown [8] that Lemma 7.3 holds for all values of $k \geq 1$, integral or not). However Turganaliev succeeds in proving (7.104) unconditionally, where $\varkappa = \varkappa(\sigma,\lambda) > 0$ is not explicitly evaluated, but depends among other things on the quality of estimates for the zero density function $N(\sigma,T)$ (see Chapter 9). The proof in the unconditional case is rather involved, where instead of using directly the estimate (7.92) (see p. 126 of Titchmarsh [8]) Turganaliev makes a careful division of points of the segment $[\sigma + iT/2, \sigma + iT]$ and makes use of Hadamard's three circles theorem and other devices to obtain (7.104) with some $\varkappa = \varkappa(\sigma,\lambda) > 0$. The range $0 < \lambda < 2$ to which Turganaliev restricts himself is motivated by the fact that the proof uses $M(4) = 1$, and as we have seen an analogous estimate of this type for $A > 4$ is still not known to hold. The method of Turganaliev can be presumably adapted to yield analogues of Theorem 7.4 when k is not an integer, with error terms as in (7.104), or perhaps only $o(T)$.

The proof of the well-known (7.100) may be found in E.C. Titchmarsh [8]. It follows from the elementary relation

$$\zeta(s) = \sum_{n\leq N} n^{-s} + N^{1-s}/(s-1) - \tfrac{1}{2}N^{-s} - s\int_N^\infty \psi(u)u^{-s-1}du$$

on letting $N \to \infty$, when one uses partial summation and Lemma 2.5 to obtain

$$\sum_{x<n\leq u} n^{-it} = \int_x^u y^{-it}dy + O(1) = \frac{u^{1-it} - x^{1-it}}{1 - it} + O(1),$$

which is used in estimating

$$\sum_{x<n\leq N} n^{-s} = \sum_{x<n\leq N} n^{-\sigma}\cdot n^{-it}.$$

The asymptotic formula (7.98) is due to Turganaliev [1], while (7.99) must be known already, although I have not been able to find a reference in the literature.

Finally we shall discuss the following interesting mean value problem, where we shall suppose that $1/2 < \sigma < 1$ and $h > 0$ are fixed. Is it true that

$$(7.105)\qquad \lim_{N\to\infty} N^{-1} \sum_{r\le N} |\zeta(\sigma + ir^h)|^2 = \zeta(2\sigma)?$$

This is a special case of a problem posed recently by A. Reich [2], where r^h is replaced by t_r for a certain natural sequence $\{t_r\}$ such as primes, square-free numbers etc. We shall sketch the proof here that (7.105) holds for $\sigma \ge 1 - g(h)$, where $g(h)$ satisfies $0 < g(h) < 1/2$. It will be sufficient to prove

$$(7.106)\qquad S_N = \sum_{N/2<r\le N} |\zeta(\sigma + ir^h)|^2 = \tfrac{1}{2}\zeta(2\sigma)N + O(N^{a(\sigma)}),\ 0 < a(\sigma) < 1.$$

Using the simple approximate functional equation (7.100) with $x = N^h$ we obtain

$$S_N = \sum_{N/2<r\le N} \Big|\sum_{n\le N^h} n^{-\sigma-ir^h}\Big|^2 + O\Big(\sum_{N/2<r\le N} \Big|\sum_{n\le N^h} n^{-\sigma-ir^h}\Big| N^{-h\sigma}\Big) + O(N^{1-2h\sigma}).$$

Since the first O-term above can be treated by the Cauchy-Schwarz inequality and

$$\sum_{N/2<r\le N} \Big|\sum_{n\le N^h} n^{-\sigma-ir^h}\Big|^2 = \tfrac{1}{2}N \sum_{n\le N^h} n^{-2\sigma} + \sum_{m\ne n\le N^h} (mn)^{-\sigma} \sum_{N/2<r\le N} (m/n)^{ir^h} =$$

(7.107)

$$\tfrac{1}{2}\zeta(2\sigma)N + O(N^{1+h-2h\sigma}) + O\Big(\sum_{1\le n<m\le N^h} (mn)^{-\sigma}\Big|\sum_{N/2<r\le N} \exp(ir^h\log m/n)\Big|\Big),$$

the problem reduces to obtaining a non-trivial bound for

$$S_N^* = S_N^*(m,n) = \sum_{N/2<r\le N} \exp(ir^h\log m/n).$$

<u>The case $0 < h < 1$</u>. This case is relatively simple. Let $f(x) = x^h\log m/n$. Then for $h > 0$ we have

$$(7.108)\qquad N^{h-1}\log m/n \ll |f'(x)| = hx^{h-1}\log m/n \ll h^2N^{h-1}\log N,$$

and $h^2N^{h-1}\log N = o(1)$ as $N \to \infty$ if $0 < h < 1$. By Lemma 2.1 and Lemma 2.1 we have

$$(7.109)\qquad S_N^* = \int_{N/2}^{N} \exp(if(x))dx + O(1) \ll \max_{N/2\le x\le N} |f'(x)|^{-1} \ll N^{1-h}(\log m/n)^{-1}.$$

Using this estimate we obtain

$$\sum_{1\le n<m\le N^h} (mn)^{-\sigma}|S_N^*(m,n)| \ll \sum_{1\le n<m\le N^h} (mn)^{-\sigma}(\log m/n)^{-1}N^{1-h} \ll$$

$$\sum_{1\le k,n\le N^h} (k+n)^{-\sigma}n^{-\sigma}\max(1,nk^{-1})N^{1-h} \ll N^{2h(1-\sigma)+1-h}$$

But for $1/2 < \sigma < 1$ one has $1 - h + 2h(1 - \sigma) < 1$, which means that for $0 < h < 1$, $1/2 < \sigma < 1$ (7.105) does indeed hold. A variation of the above argument shows that this is also true for $h = 1$, a fact which also follows from Satz 2.3 of A. Reich [1].

The case $h > 1$. Suppose first that h is not an integer. Considering the range $m > 2n$ in S_N^* we may in view of (7.108) use the theory of exponent pairs and deduce that

(7.110) $$S_N^*(m,n) \ll N^{p(h-1)+q}\log N, \quad (m > 2n)$$

for any exponent pair (p,q). For $n < m \le 2n$ we estimate S_N^* either by (7.110) if $f'(x) \gg 1$ or by (7.109) otherwise, which again easily leads to (7.105). Then from (7.107) and (7.110) it is seen that (7.105) follows if

(7.111) $$p(h - 1) + q < 1$$

is satisfied. Recalling that for $L = 2^{l-1}$, $l \ge 3$

(7.112) $$(p,q) = (1/(2L - 2), (2L - l - 1)/(2L - 2))$$

is an exponent pair (see the discussion after (6.60)), we choose in (7.12) $l = [h] + 1$. Then (7.111) holds and (7.105) follows for $\sigma \ge 1 - g(h)$, $0 < g(h) < 1/2$, as asserted. One may evaluate g(h) explicitly, e.g. for $h = 3/2$ with $(p,q) = (2/7, 4/7)$ we obtain that (7.105) holds for $\sigma > 19/21$. Since the theory of exponent pairs can be in fact built from the knowledge of the first four derivatives of the function in question, the above discussion covers all the cases except $h = 2$ and $h = 3$. For $h = 2$ we square out directly $|S_N^*|^2$, grouping together suitable terms and using rational approximations to $\pi^{-1}\log m/n$. For $h = 3$ one may use Lemma 2.7 to transform S_N^* into an exponential sum of the same type (plus manageable error terms), corresponding essentially to the original sum for $h = 3/2$, and the latter sum can be dealt with by using the theory of exponent pairs.

Various generalizations of (7.105) are possible. For instance, one may pose the corresponding problem when the square of the modulus is replaced by an arbitrary, fixed even power etc.

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TOPICS IN RECENT ZETA-FUNCTION THEORY

CHAPTER 8

CONSECUTIVE ZEROS ON THE CRITICAL LINE

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CONSECUTIVE ZEROS ON THE CRITICAL LINE

§1. Introduction

This chapter is devoted to the study of consecutive zeros of the form $1/2 + it$, where t is real and positive. A classical result of A. Selberg states that there are $\gg T\log T$ of these zeros for $0 < t \leq T$, and quite a few have been calculated numerically. In view of Riemann's hypothesis it is only natural that zeros on the critical line $1/2 + it$ have always attracted much attention. If γ_n is the imaginary part of the n-th zero of the zeta-function on the line $1/2 + it$, then one of the most interesting and yet unsettled problems concerning these zeros is the estimation of the gap between consecutive zeros, i.e. inequalities of the type

$$\gamma_{n+1} - \gamma_n \ll \gamma_n^c \log^d \gamma_n \tag{8.1}$$

with some $0 \leq c < 1$ and $d \geq 0$. The first result in this direction is the classical theorem of G.H. Hardy and J.E. Littlewood [1] that (8.1) holds with $c = 1/4 + \varepsilon$. Their investigations were based on the properties of the function

$$Z(t) = \chi^{-1/2}(1/2 + it)\zeta(1/2 + it), \tag{8.2}$$

where χ is the function defined by the functional equation (4.3). The result of Hardy and Littlewood remained the best one for an exceptionally long time, until independently J. Moser [1], [2] and R. Balasubramanian [1] proved that (8.1) holds with $c = 1/6$, $d = 5 + \varepsilon$ and $c = 1/6 + \varepsilon$ respectively. Their methods of proof were different, and Moser's method forms part of his extensive study of properties of the function $Z(t)$; he also obtained in [2] the conditional résult $c = 1/8 + \varepsilon$ in (8.1) if the Lindelöf hypothesis that $\zeta(1/2 + it) \ll |t|^\varepsilon$ is true. Balasubramanian's approach stems from his work [1] on the mean square of the zeta-function on the critical line and necessitates a lower bound for $\int_{T-H}^{T+H} |\zeta(1/2 + it)|dt$. As mentioned in Notes of Chapter 7, K. Ramachandra [4], [5] obtained general lower bounds of integrals of certain Dirichlet series, and in particular he showed that for $k \geq 1$ a fixed integer

$$(8.3)\qquad \int_{T-H}^{T+H} |\zeta(1/2+it)|^k dt \gg H(\log T)^{k^2/4},\ T^{\epsilon} \le H \le T,$$

where the $\ll$-constants depend on k only.

By any of the methods used in the above mentioned works it is seen that the problem of obtaining (8.1) reduces to the estimation of a "short" exponential sum after some averaging process. Following Moser's method of approach and taking into account the particular structure of the exponential sum in question, A.A. Karacuba [4] recently obtained (8.1) with $c = 5/32$, $d = 2$. The exponent $5/32$ is interesting, since it is smaller than the best known exponent $35/216$ of G. Kolesnik (see (6.63)) for the order of $\zeta(1/2 + it)$. Our purpose in this chapter is to prove the following

<u>THEOREM 8.1</u>. For any $\epsilon > 0$ and $n \ge n_0(\epsilon)$,

$$(8.4)\qquad \gamma_{n+1} - \gamma_n \ll \gamma_n^{u+\epsilon},\ u = 0.1559458\dots .$$

This clearly improves the exponent $5/32$ of Karacuba, since $5/32 = 0.15625$. Instead of using what R. Balasubramanian [1] has called "the multiple integration process", we shall reduce the problem to the estimation of a short exponential sum by smoothing with the exponential integral (1.34). Further, instead of using the sharp bound (8.3) with $k = 1$, whose proof is rather involved, for our purposes it will be sufficient to use the weaker

<u>Lemma 8.1</u>. For any $k \ge 1$ a fixed integer, $T^{\epsilon} \le H \le T$, we have uniformly in H

$$(8.5)\qquad \int_{T-H}^{T+H} |\zeta(1/2+it)|^k dt \gg H(\log T)^{-1/2-\epsilon}.$$

Lemma 8.1 will be proved in §2 very simply by using also the exponential integral (1.34), while a discussion of estimates (8.1) and a proof of (8.4) will be given in §3.

§2. <u>Proof of Lemma 8.1</u>

Let $G = H(\log T)^{-1/2-\epsilon}$ and

$$(8.6)\qquad I_k = \left|\int_{-H}^{H} \zeta^k(1/2+it+iT)e^{-t^2G^{-2}}dt\right| \le \int_{-H}^{H} |\zeta(1/2+it+iT)|^k e^{-t^2G^{-2}}dt$$

$$\le \int_{T-H}^{T+H} |\zeta(1/2+it)|^k dt.$$

Then by Cauchy's integral theorem we have

$$I_k = \left| \int_{\frac{1}{2}+i(T-H)}^{\frac{1}{2}+i(T+H)} \zeta^k(s) e^{(s-1/2-iT)^2 G^{-2}} ds \right| =$$

$$= o(1) + \left| \int_{2+i(T-H)}^{2+i(T+H)} \zeta^k(s) e^{(s-1/2-iT)^2 G^{-2}} ds \right|,$$

since the choice $G = H(\log T)^{-1/2-\varepsilon}$ makes the integrals over the segments $\sigma + i(T \pm H)$, $1/2 \le \sigma \le 2$ equal to $o(1)$, because trivially $\zeta(\sigma + it) \ll t^{1/6}$ for $\sigma \ge 1/2$ and $\exp(-C_1 H^2 G^{-2}) \ll T^{-C_2}$ for any fixed $C_1, C_2 > 0$. The same argument shows that the integral over $[2 + i(T \pm H), 2 \pm i\infty)$ is $o(1)$, so that using (1.34) we have

$$I_k = \left| \int_{-\infty}^{\infty} \zeta^k(2 + it + iT) e^{(3/2+it)^2 G^{-2}} dt \right| + o(1) =$$

$$\left| \sum_{n=1}^{\infty} d_k(n) n^{-2-iT} \int_{-\infty}^{\infty} n^{-it} e^{(9/4+3it-t^2)G^{-2}} dt \right| + o(1) = \tag{8.7}$$

$$\pi^{1/2} G + \pi^{1/2} G \exp(\tfrac{9}{4} G^{-2}) \sum_{n=2}^{\infty} d_k(n) n^{-2-iT} \exp(-\tfrac{1}{4} G^2 (3G^{-2} - \log n)^2) + o(1) =$$

$$\pi^{1/2} G + o(1) \ge G,$$

since the series for $\zeta^k(2 + it + iT)$ converges absolutely and may be integrated termwise, and for any $n \ge 2$, any fixed $C > 0$ and T sufficiently large

$$\exp(-\tfrac{1}{4} G^2 (3G^{-2} - \log n)^2) \le \exp(-(\tfrac{1}{3} \log 2 \cdot G)^2) \ll T^{-C}.$$

Lemma 8.1 follows therefore from (8.6) and (8.7).

§3. Proof of Theorem 8.1.

As stated in §1 we shall smoothen exponantial sums that appear in the carse of the proof with the aid of (1.34). This is considerably simpler than averaging with multi-dimensional processes, and when combined with (8.3) (k = 1) produces a better log-factor in (8.1) than does Balasubramanian's or Moser's method if one uses the same van der Corput estimates for exponential sums. However this in itself is not sufficient to reduce the exponent $c = 5/32$ in (8.1) to the one given by (8.4), and to obtain this improvement we shall employ the same idea as was done by A.A. Karacuba in [4], only the short exponential sum to which the problem reduces will be estimated more carefully with the theory of exponent pairs. By using properties of exponent pairs developed in §3 of Chapter 2, it will be seen that the value of u in (8.4) is in fact the best one this method allows. For possible slight sharpenings of Theorem 8.1 one would thus have to turn to two- and multi-dimensional methods of estimating exponential sums.

The idea of proof of Theorem 8.1, which may be traced to Hardy's classic proof that there are infinitely many zeros of the zeta-function on the critical line, is to suppose that $Z(t)$ (given by (8.2)) does not change sign in $[T-U, T+U]$ and to show that this is impossible with a suitably chosen $U = U(T) = o(T)$. Thus we suppose that $|I_1| = I_2$, where

$$I_1 = \int_{T-U}^{T+U} \exp(-(T-u)^2 U^{-2} L)\cdot Z(u)\,du, \tag{8.8}$$

$$I_2 = \int_{T-U}^{T+U} \exp(-(T-u)^2 U^{-2} L)\cdot |Z(u)|\,du, \tag{8.9}$$

where for convenience we shall use the notation $L = (\log T)^{1+\varepsilon}$. Then by Lemma 8.1 with $k = 1$ and $U > L$,

$$I_2 \geq \int_{T-UL^{-1/2}}^{T+UL^{-1/2}} \exp(-(T-u)^2U^{-2}L)|\zeta(1/2 + iu)|\,du \geq \tag{8.10}$$

$$\geq e^{-1}\int_{T-UL^{-1/2}}^{T+UL^{-1/2}} |\zeta(1/2 + iu)|\,du \gg U(\log T)^{-1-\varepsilon}.$$

We want to majorize I_1 in (8.8), and the simplest way to do this seems to be the use of the approximate functional equation (4.12), which gives with the abbreviation $Q = (T/2\pi)^{1/2}$

$$I_1 = \int_{-U}^{U} \exp(-u^2U^{-2}L)\chi^{-1/2}(1/2 + iT + iu)\zeta(1/2 + iT + iu)\,du = \tag{8.11}$$

$$\overline{S}_1 + S_1 + O(U^2T^{-1/4}),$$

where

$$S_1 = \sum_{n\leq Q} n^{-1/2+iT}\int_{-U}^{U}\chi^{1/2}(1/2 + iT + iu)n^{iu}\exp(-u^2U^{-2}L)\,du. \tag{8.12}$$

Now we recall the asymptotic formula (4.4) for $\chi(s)$ and abbreviate

$$f(x) = \frac{x}{2}\log(2\pi/x) + \frac{x}{2} + x\log n + \frac{\pi}{8}.$$

Using Taylor's formula and $|\exp(ix) - \exp(iy)| \le |x - y|$ (x,y real) we obtain

$$(8.13) \quad S_1 = \sum_{n\le Q} n^{-1/2}\exp(if(T))\int_{-\infty}^{\infty}\exp(iu\log(n/Q) - \frac{iu^2}{4T} - u^2U^{-2}L)\,du + O(U^4T^{-7/4}) + O(UT^{-3/4}),$$

since $\int_{\pm U}^{\pm\infty} \ll \exp(-L/2) \ll T^{-c}$ for any fixed $c > 0$. At this point we restrict U to the range $T^{\varepsilon} \le U \le T^{1/3}$, and setting $X = (4T/i)^{-1} + U^{-2}L$ we obtain

$$X^{-1/2}\exp(-(4X)^{-1}\log^2(n/Q)) = UL^{-1/2}\exp(-U^2(4L)^{-1}\log^2(n/Q)) + O(U^3T^{-1}).$$

Therefore using (1.34) it follows from (8.13) that

$$(8.14) \quad S_1 = \pi^{1/2}UL^{-1/2}\sum_{n\le Q} n^{-1/2}\exp(if(T))\exp(-U^2(4L)^{-1}\log^2(n/Q)) + O(U^3T^{-3/4}).$$

Although the sum that appears in (8.14) has many terms, the presence of the second exponential factor will make the contribution of many of these terms negligible. To see this we let $P = [Q] = [(T/2\pi)^{1/2}]$, $n = P - m$, where m is an integer satisfying

$$m > QU^{-1}L^{1+\varepsilon} = [(2\pi)^{-1/2}T]U^{-1}(\log T)^{(1+\varepsilon)^2}.$$

But we have

$$U^2L^{-1}\log^2(n/Q) = U^2L^{-1}\left\{\log(1 - (m + O(1))Q^{-1}\right\}^2 \ge \frac{1}{2}L^{1+\varepsilon},$$

and the second exponential factor in (8.14) makes the contribution of these n to S_1 negligible. For the remaining n in (8.14) we obtain by partial summation

$$(8.15) \quad I_1 \ll U^2T^{-1/4} + UT^{-1/4}L^{-1/2}\sum_M \max_{M'}\Big|\sum_{M<m\le M'\le 2M}\exp(iT\log(P - m))\Big|,$$

where $P = [(T/2\pi)^{1/2}]$, the maximum is taken over M' satisfying $M < M' \le 2M$, and $\sum_M$ denotes summation over $O(\log T)$ values $M = 2^{-j}QU^{-1}L^{1+\varepsilon}$, $j = 1,2,\ldots$, so that the exponential sums in (8.15) are short in the sense that $M = o(P)$. To see first how one obtains the value $c = 1/6$ in (8.1) of Balasubramanian and Moser, we use the classical van der Corput estimate

$$(8.16) \quad \sum_{a<n\le b} e^{if(n)} \ll (b - a)\lambda_3^{1/6} + (b - a)^{1/2}\lambda_3^{-1/6},$$

where $f(x)$ is real, $b - a \geq 1$, $\lambda_3 \asymp |f^{(3)}(x)|$ for $a \leq x \leq b$. Applying (8.16) with $M = a$, $M' = b$, $f(x) = T\log(P - x)$, $\lambda_3 = T^{-1/2}$ and summing over M we obtain on comparing (8.15) with (8.10)

$$U(\log T)^{-1-\varepsilon} \ll I_2 = |I_1| \ll U^2T^{-1/4} + T^{1/6}(\log T)^{1/2+\varepsilon} + U^{1/2}T^{1/12}\log^{\varepsilon}T,$$

provided that $T^{\varepsilon} \leq U \leq T^{1/3}$, and this is impossible with $U = T^{1/6}\log^{2+\varepsilon}T$. Therefore $Z(t)$, and consequently $\zeta(1/2 + it)$ must have a zero in $[T - U, T + U]$ with this choice of U. Setting $\gamma_n = T - U$ we have $\gamma_{n+1} \in [T - U, T + U]$, hence (8.1) with $c = 1/6$, $d = 2 + \varepsilon$. Using the stronger (8.3) instead of (8.5) we would only gain on the quality of the log-factor and obtain (8.1) with $c = 1/6$, $d = 3/4 + \varepsilon$. However (8.3) is much more difficult to prove than (8.5), and the latter estimate is sufficient for (8.4) which contains ε in the exponent.

Therefore to obtain the exponent given by Theorem 8.1 we have to estimate more carefully the sum

$$\text{(8.17)} \qquad S = S(M, M', T) = \sum_{M<m\leq M'\leq 2M} \exp(iT\log(P - m)),\ P = [(T/2\pi)^{1/2}],\ M \ll T^{1/2}U^{-1}L^{1+\varepsilon}.$$

From the definition of P we have $T = 2\pi(P + \theta)^2$ for some $0 \leq \theta < 1$, and therefore

$$T\log(P - m) - T\log P = -T\sum_{k=1}^{\infty}(m/P)^k k^{-1} =$$

$$-2\pi Pm - 2\pi(2P\theta + \theta^2)mP^{-1} - \pi m^2 - 2\pi(2P\theta + \theta^2)m^2(2P^2)^{-1} -$$

$$-T(m^3/(3P^3) + m^4/(4P^4) + \ldots).$$

Taking into account that $\exp(2\pi ir) = 1$ for any integer r we consider separately even and odd m (to get rid of πm^2) to obtain

$$|S| \leq |S'| + |S''|,$$

where S' comes from even m and equals

$$\text{(8.18)} \qquad S' = \sum_{M_1<m\leq M_1'\leq 2M_1} \exp(2\pi iF(m)),\ M \ll M_1 \ll M,$$

$$\text{(8.19)} \qquad F(x) = c_1x + c_2x^2 + T(2\pi)^{-1}((2x)^3/(3P^3) + (2x)^4/(4P^4) + \ldots),$$

$$c_1 = 2(2\theta P + \theta^2)P^{-1} = O(1),\ c_2 = c_1P^{-1} = O(P^{-1}).$$

The expression for S" (coming from odd m) is similar and hence it will be sufficient to estimate S' in (8.18). For $M \gg T^{1/4}$ and $M_1 \le x \le 2M_1$ we have $|F'(x)| \gg 1$ and

$$|F^{(k)}(x)| \asymp M^{3-k}T^{-1/2}, k \le 3; \quad |F^{(k)}(x)| \asymp T^{1-k/2} < M^{3-k}T^{-1/2}, \quad k > 3,$$

where the $\asymp$-constants depend only on k. This means that we may estimate S' by the theory of exponent pairs of Chapter 2 as

$$S' \ll A^p M^q, \tag{8.20}$$

where (p,q) is any exponent pair and

$$A = \max_{M_1 \le x \le 2M_1} |F'(x)| \ll M^2 T^{-1/2}. \tag{8.21}$$

Thus for $M \gg T^{1/4}$ we use (8.20) and (8.21), and for $M \ll T^{1/4}$ we use (8.16) to estimate S'. Then we obtain

$$S \ll M^{2p+q}T^{-p/2} + T^{5/24}. \tag{8.22}$$

Summing over various M and keeping in mind that $M \ll T^{1/2}U^{-1}L^{1+\epsilon}$ we obtain from (8.15)

$$I_1 \ll U^2T^{-1/4} + UT^{-1/24}\log T + UT^{(p+q)/2-1/4}U^{-(2p+q)}(\log T)^{2p+q-1/2+\epsilon}. \tag{8.23}$$

Comparing this estimate with (8.10) as before we obtain a contradiction of $|I_1| = I_2$ if $T^{\epsilon} \le U \le T^{1/4}\log^{-1}T$ and

$$U = T^{(2p+2q-1)/4(2p+q)}(\log T)^{(4p+2q+1+\epsilon)/2(2p+q)}, \tag{8.24}$$

and if we used (8.3) with k = 1 instead of (8.5) we would obtain the slightly better exponent $(8p + 4q - 1 + \epsilon)/4(2p + q)$ of logT in (8.24), but as already mentioned this is of no importance for the proof of (8.4). Now we use Lemma 2.8 to deduce that if (p,q) is an exponent pair then $(p/(2p+2), 1/2 + q/(2p+2))$ is also an exponent pair, and further an application of Lemma 2.9 shows that $(q/(2p+2), (2p+1)/(2p+2))$ is an exponent pair too. Replacing (p,q) by this last exponent pair in (8.24) we obtain the condition

$$U = T^{(p+q)/2(2p+2q+1)}(\log T)^{(3p+2q+2+\epsilon)/(2p+2q+1)} \tag{8.25}$$

for which $|I_1| = I_2$ is falsified, and this choice of U trivially satisfies

$T^{\varepsilon} \le U \le T^{1/4}\log^{-1}T$. The purpose of this transformation is that now the exponent of T in (8.25) is an increasing function of p + q, which means that the best result will be obtained if we take for (p,q) the exponent pair for which p + q is a minimum. In accordance with the discussion made at the end of Chapter 2 we take $(p,q) = (\alpha/2 + \varepsilon, 1/2 + \alpha/2 + \varepsilon), \alpha = 0.3290213568...$, which is then precisely the exponent pair for which p + q is minimal, and then

$$(p + q)/2(2p + 2q + 1) = 0.15594583...,$$

which proves (8.4). It may be noted that the trivial exponent pair (p,q) = (0,1) in (8.25) leads to c = 1/6 in (8.1), while Karacuba's value c = 5/32 follows from the standard exponent pair (p,q) = (1/6,2/3).

NOTES

A. Selberg's classical result that

$$(8.26) \qquad N_0(T) > \frac{CT\log T}{2\pi}$$

for some absolute,unspecified C > 0 is given in Chapter 10 of Titchmarsh [8]. Here $N_0(T)$ denotes the number of zeros of $\zeta(s)$ of the form $s = 1/2 + it$, $0 < t \le T$. If N(T) denotes the number of zeros of the zeta-function in the rectangle $0 \le \sigma \le 1$, $0 \le t \le T$, then the classical formula of Riemann-von Mangoldt states that (chapter 9 of Titchmarsh [8])

$$(8.27) \qquad N(T) = (2\pi)^{-1}T\log T - (1 + (2\pi)^{-1})T + O(\log T),$$

which implies that up to the value of C the estimate (8.26) of Selberg is best possible. N. Levinson [1] obtained $C \ge 1/3$ in (8.26), which is a remarkable achievement, and his result has been recently improved byShi-Tuo Lou [1] to $C \ge 0.35$ by an elaboration of Levinson's method. Also by adapting Levinson's method D.R. Heath-Brown [5] proved that more than one third of zeros of $\zeta(s)$ are simple and on the critical line, but in spite of the quality of this type of results it is rather improbable that Levinson's method will ever give $N_0(T) \sim N(T)$ as $T \to \infty$.

The Riemann hypothesis and (8.27) trivially imply

$$\gamma_{n+1} - \gamma_n \ll \log\gamma_n,$$

but it would be interesting to investigate whether anything can be deduced from inequalities of the type (8.1) about the horizontal distribution of zeros of the zeta function.

As mentioned in the preface, Theorem 8.1 is due to the author and is hitherto unpublished.

With the advent of refined computational techniques calculations concerning zeros on the critical line have been carried out to a remarkable extent. A detailed account of these techniques may be found in the book of H.M. Edwards [1] and in the paper by J.B. Rosser et al. [1], where additional references may be found. The best result at the time of writing of this text seems to be that of R.P. Brent [1], who obtained by calculations that the first 75 000 001 zeros of the zeta-function which are complex are simple and lie on the critical line (the imaginary parts of the first six zeros are 14.13, 21.02, 25.01, 30.42, 32.93, 37.58 approximately). No doubt numerical data will continue to accrue rapidly.

A proof of Lemma 8.1 with the right-hand side of (8.5) replaced by H follows from Theorem 3 of R. Balasubramanian [1]. The proof given there is due to K. Ramachandra, who has also sharper and more extensive results on this subject of which some were discussed in the Notes of Chapter 7.

The discrete method of E.C. Titchmarsh in zeta-function theory (see Chapter 2 of Titchmarsh [8] for outlines of the method) deals with sums involving the sequence $\{t_\nu\}_{\nu=1}^{\infty}$, where t_ν is the unique real root of

$$\vartheta(t) = \pi\nu \quad (\nu \geq \nu_0 > 0), \quad Z(t) = e^{i\vartheta(t)}\zeta(1/2 + it),$$

and $Z(t)$ is defined by (8.2). Therefore

$$\vartheta(t) = -\frac{1}{2}t\log\pi + \operatorname{Im}\log\Gamma\left(\frac{1}{4} + \frac{1}{2}it\right) = \frac{1}{2}t\log(t/2\pi) - \frac{t}{2} - \frac{\pi}{8} + O(1/t),$$

$$\vartheta'(t) = \frac{1}{2}\log(t/2\pi) + O(1/t).$$

E.C. Titchmarsh [4] proved

$$\sum_{\nu=M+1}^{N} Z(t_\nu)Z(t_{\nu+1}) \sim -2(\gamma + 1)N, \tag{8.28}$$

where $M > 0$ is fixed and sufficiently large, and he conjectured

$$(8.29)\qquad \sum_{\nu=M+1}^{N} Z^2(t_\nu)Z^2(t_{\nu+1}) = O(N\log^A N)$$

for some $A \geq 0$. It may be noted that if for some t_ν^* we have $Z(t_\nu^*)Z(t_{\nu+1}^*) < 0$, then there is a zero $1/2 + it$ of the zeta-function satisfying $t \in [t_\nu^*, t_{\nu+1}^*]$, so that from (8.28) one immediately has as a corollary Hardy's classical result that there are infinitely many zeros on the critical line. This fact shows the connection between the sequence $\{t_\nu\}_{\nu=1}^{\infty}$ and zeros on the critical line.

In a series of papers (some of which are [1], [2], [3], [4]) J. Moser obtained several interesting results concerning Titchmarsh's discrete method, and in particular in [4] he proved Titchmarsh's conjecture (8.29) with $A = 4$. From the Cauchy-Schwarz inequality and $\zeta(1/2 + it) \ll t^{1/6}$ it is seen that (8.29) will follow from

$$(8.30)\qquad \sum_{\nu=M+1}^{N} Z^4(t_\nu) \ll N\log^A N.$$

Taking into account that $|\zeta(1/2 + it)| = |Z(t)|$ it is clear that (8.30) is a consequence of the discrete fourth power moment estimate

$$(8.31)\qquad \sum_{r\leq R} |\zeta(1/2 + it_r^*)|^4 \ll T\log^5 T,\quad |t_r^*| \leq T,\quad |t_r^* - t_s^*| \geq 1 \quad \text{for } r \neq s \leq R,$$

which follows from (7.6). Namely

$$t_{\nu+1} - t_\nu \sim 2\pi(\log t_\nu)^{-1},$$

so that each interval $[t, t + 1]$ contains $\ll \log t$ numbers t_ν, and then we may define t_r^* by

$$|\zeta(1/2 + it_r^*)| = \max_{r\leq t_\nu\leq r+1} |Z(t_\nu)|,\quad r = 1,2,\ldots .$$

Considering separately t_{2m}^* and t_{2m+1}^* we have the spacing condition $|t_r^* - t_s^*| \geq 1$ for $r \neq s$, and since $N \sim (2\pi)^{-1}T\log T$ if $\nu \leq N$ and $t_\nu \leq T$, then collecting $O(\log T)$ estimates of the type (8.31) we obtain (8.30) with $A = 5$, and consequently (8.29) too. This is poorer by a log-factor than Moser's result, but the derivation sketched here is much simpler. Higher power moments of Chapter 7 allow one to estimate in a similar way sums of the type

$$\sum_{\nu=M+1}^{N} Z^{2k}(t_\nu)Z^{2k}(t_{\nu+1}),$$

where $k \geq 1$ is a fixed integer.

The estimate (8.16) of van der Corput is standard and was not proved in Chapter 2, since it was not needed before and besides it is given as Theorem 5.11 of Titchmarsh [8]. Its proof follows from (2.38), when each sum

$$\sum_{a<n\leq b-h} e(f(n+h)-f(n))$$

is estimated by

$$\sum_{X<n\leq Y} e(F(n)) \ll (Y - X)\lambda_2^{1/2} + \lambda_2^{-1/2}, \tag{8.32}$$

where $|F''(x)| \geq \lambda_2$ for $X \leq x \leq Y$, and H is then chosen optimally. The estimate (8.32) is an easy consequence of Lemma 2.2 and Lemma 2.4, since $e(F(n)) = e(F(n)-kn)$ for any integer k. Thus one may split the sum in (8.32) into not more than

$$|F'(Y) - F'(X)| + 1 \ll (Y - X)\lambda_2 + 1$$

subsums, and to each of these Lemma 2.5 is applied and the integral estimated by (2.2) to produce (8.32).

A.A. Karacuba's paper [4] contains also a result on zeros of $Z^{(k)}(t)$ in short intervals. If $k \geq 1$ is a fixed integer, $T \geq T_0$,

$$H \gg T^{1/(6k+6)}(\log T)^{2/(k+1)},$$

then Karacuba proves that every interval $(T, T + H]$ contains a zero of $Z^{(k)}(t)$ of odd order. The main tool in the proof of this result is an approximate functional equation for $Z^{(k)}(t)$, similar to the approximate functional equation (4.12), where the length of the Dirichlet polynomials approximating $Z^{(k)}(t)$ is $(t/2\pi)^{1/2}$ and the error term is $O(t^{-1/4}\log^k t)$.

Aleksandar Ivić

TOPICS IN RECENT ZETA-FUNCTION THEORY

C H A P T E R 9

Z E R O-D E N S I T Y E S T I M A T E S

CHAPTER 9

ZERO-DENSITY ESTIMATES

§1. Introduction

Zero-density estimates involve upper bounds for the function $N(\sigma,T)$, which represents the number of zeros $\rho = \beta + i\gamma$ (β,γ real) of the zeta-function for which $\beta \geq \sigma \geq 0$, where σ is fixed and $-T \leq \gamma \leq T$. Estimates for $N(\sigma,T)$ may be written in the form

$$(9.1) \qquad N(\sigma,T) \ll T^{A(\sigma)(1-\sigma)}\log^C T$$

where $C \geq 0$, or

$$(9.2) \qquad N(\sigma,T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon},$$

where we shall always suppose that the $\ll$-constant is uniform in σ and T, but depends only on ε. In view of the Riemann-von Mangoldt formula (8.27) one has trivially $A(\sigma)(1-\sigma) = 1$, $C = 1$ in (9.1) for $0 \leq \sigma \leq 1/2$, while for $\sigma > 1/2$ obviously $A(\sigma)(1-\sigma) \leq 1$ and $A(\sigma)(1-\sigma)$ is non-increasing. Zero-density estimates have a large number of applications in many branches of analytic number theory, and it turns out that in some of these applications (like the problem of the estimation of the difference between consecutive primes) results obtainable from the Lindelöf (or even Riemann) hypothesis follow in almost the same degree of sharpness from a much weaker conjecture, namely

$$(9.3) \qquad A(\sigma) \leq 2, \quad 1/2 \leq \sigma \leq 1,$$

which (both in (9.1) and in (9.2)) is known as "the density hypothesis". As for many applications (9.1) does not have much advantage over the somewhat weaker (9.2), we shall be concerned mostly with estimates of the type (9.2), formulating our results for convenience as upper bounds for $A(\sigma)$ in (9.2) rather than for $N(\sigma,T)$ itself. In view of the preceding discussion and the well-known fact that there are no zeros on the line $\sigma = 1$, it is sufficient to consider the range $1/2 < \sigma < 1$ in (9.2). Except when σ is very close to 1/2 or 1 we shall prove in this chapter the sharpest known bounds for $A(\sigma)$. To accomplish this we shall use a zero-detection method, which will be fully explained in §2, and which offers great flexibility in the estimation of $A(\sigma)$. Among other tools we shall use the higher power moment estimates of Chapter 7, and certain double zeta sums, which will be considered in §6 and §8.

§2. The zero-detection method

We start from (1.7) with $x = n/Y$, namely

$$e^{-n/Y} = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty}\Gamma(w)Y^w n^{-w}dw, \tag{9.4}$$

and let $M_X(s) = \sum_{n\le X}\mu(n)n^{-s}$, where $s = \sigma + it$, $\log^2 T \le |t| \le T$, $1 \ll X \le Y \ll T^C$, $\mu(n)$ is the Möbius function and $X = X(T)$, $Y = Y(T)$ are parameters which will be suitably chosen. In view of the elementary relation

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases},$$

it is seen that each zero $\rho = \beta + i\gamma$ of $\zeta(s)$ counted by $N(\sigma,T)$ satisfies

$$e^{-1/Y} + \sum_{n>X}a(n)n^{-\rho}e^{-n/Y} = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty}\zeta(\rho + w)M_X(\rho + w)Y^w\Gamma(w)dw, \tag{9.5}$$

where

$$a(n) = \sum_{d|n,d\le X}\mu(d), \quad |a(n)| \le d(n) < n^\epsilon, \tag{9.6}$$

since the absolutely convergent series $\zeta(\rho + w) = \sum_{n=1}^{\infty}n^{-\rho-w}$ and $M_X(\rho + w)$ may be multiplied and then integrated termwise using (9.4).

Now the line of integration in (9.5) is moved to $\mathrm{Re}\, w = 1/2 - \beta$. For $|\gamma| \ge \log^2 T$ the residue at the pole $w = 1 - \rho$ of the integrand is $o(1)$ by (1.32), and the pole $w = 0$ of $\Gamma(w)$ is cancelled by the zero $w = 0$ of $\zeta(\rho + w)$. Also using (1.32) one has

$$\int_{\mathrm{Re}w=1/2-\beta} = o(1) + (2\pi i)^{-1}\int_{-\log^2 T}^{\log^2 T}\zeta(1/2+i\gamma+iv)M_X(1/2+i\gamma+iv)\Gamma(1/2-\beta+iv)Y^{\frac{1}{2}-\beta+iv}dv, \tag{9.7}$$

and also one has trivially

$$\sum_{n>Y\log^2 Y}a(n)n^{-\rho}e^{-n/Y} = o(1) \tag{9.8}$$

as $Y\to\infty$. But then $\exp(-1/Y)\to 1$, so that each $\rho = \beta + i\gamma$ counted by $N(\sigma,T)$ satisfies at least one of the following conditions:

(9.9) $$\sum_{X<n\le Y\log^2 Y} a(n)n^{-\rho}e^{-n/Y} \gg 1,$$

(9.10) $$\int_{-\log^2 T}^{\log^2 T} \zeta(1/2+i\gamma+iv)M_X(1/2+i\gamma+iv)\Gamma(1/2-\beta+iv)Y^{1/2-\beta+iv}dv \gg 1,$$

(9.11) $$|\gamma| \le \log^2 T.$$

The number of zeros ρ satisfying (9.11) is trivially $O(\log^3 T)$, since each strip $T \le t \le T+1$ contains $O(\log T)$ zeros by the Riemann-von Mangoldt formula (8.27). By the same argument we may choose R_1 zeros satisfying (9.9) and R_2 zeros satisfying (9.10) so that the imaginary parts of these zeros differ from each other by at least $2\log^4 T$ and therefore

(9.12) $$N(\sigma,T) \ll (R_1 + R_2 + 1)\log^5 T.$$

At this point we choose simply

(9.13) $$X = T^{\varepsilon}$$

so that trivially

(9.14) $$M_X(1/2 + i\gamma + iv) \ll T^{\varepsilon} \quad \text{for} \quad |v| \le \log^2 T.$$

Next we regulate the length of the Dirichlet polynomial appearing in (9.9) by observing that each ρ counted by R_1 satisfies

(9.15) $$\sum_{N<n\le 2N} a(n)n^{-\rho}e^{-n/Y} \gg 1/\log Y$$

for at least one of $O(\log Y)$ values $T^{\varepsilon} \le N = 2^{-j}Y\log^2 Y$, $j = 1,2,\ldots$, and we may consider representative zeros of those counted by R_1 which are $\gg R_1/\log Y$ in number and which satisfy (9.15) with a particular N. The exact size of N is not important, since we are going to raise (9.15) to the power k, where k is a natural number depending on N such that $N^k = M$, $(2N)^k = P \le T^C$, whence $k \ll 1$ and

$$\sum_{M<n\le P} b(n)n^{-\rho} \gg 1/\log^k Y$$

with $b(n) \ll d_{2k}(n) \ll n^{\varepsilon}$ and $P \ll M$. We split this last sum into subsums of length not exceeding M and choose k so that $N^k \le Y^r\log^{2r} Y < N^{k+1}$, $k \ge r \ge 2$ is satisfied, where r is a fixed integer. Then we have

(9.16) $$Y^{r^2/(r+1)}\log^{2r^2/(r+1)} Y \ll M \ll Y^r\log^{2r} Y,$$

and in view of existing power moments for the zeta-function it turns out that the practical choice for r in (9.16) is $r = 2$, which gives

$$Y^{4/3}\log^{8/3}Y \ll M \ll Y^2\log^4 Y. \tag{9.17}$$

Therefore we have reduced the estimation of R_1 to the estimation of the number of representative zeros $\rho = \sigma + i\gamma, \sigma \geq \delta$ of R_1 for which

$$\sum_{M<n\leq 2M} b(n)n^{-\delta-i\gamma} \gg 1/\log^D T \tag{9.18}$$

for some $1 \ll D \ll 1$, $b(n) \ll n^{\varepsilon}$ and M satisfying (9.17), since by the partial summation formula (1.17) we may replace $n^{-\rho}$ by $n^{-\delta-i\gamma}$ without affecting (9.17) and the order of magnitude of the b(n)'s (in the sense that they remain $\ll n^{\varepsilon}$).

To estimate R_2 we set for $r = 1,\dots,R_2$

$$|\zeta(1/2 + i\gamma_r + iv')| = \max_{-\log^2 T\leq v\leq \log^2 T} |\zeta(1/2 + i\gamma_r + iv)|$$

and

$$t_r = \gamma_r + v',$$

where $\gamma_1,\dots,\gamma_{R_2}$ are ordinates of zeros satisfying (9.10), and then from (9.10) we infer that

$$1 \ll T^{\varepsilon}Y^{1/2-\delta}|\zeta(1/2 + it_r)|, \quad r = 1,\dots,R_2. \tag{9.19}$$

For $r \neq s$ obviously $|t_r - t_s| \geq \log^4 T$, and so raising (9.19) to the power $A \geq 4$ we have

$$R_2 \ll T^{\varepsilon}\sum_{r\leq R_2} |\zeta(1/2 + it_r)|^A Y^{A(1/2-\delta)} \ll T^{M(A)+\varepsilon}Y^{A(1/2-\delta)}, \tag{9.20}$$

where M(A) is defined by (7.1). We may also utilize Theorem 7.1 to estimate R_2. Defining $H(T) = T^{q/(1-2p+4q)}$ it is seen that (7.8) gives

$$R \ll \begin{cases} T^{1+\varepsilon}V^{-6}, & V \geq H(T) \\ T^{(p+q+\varepsilon)/p}V^{-2(1+2p+2q)/p}, & V \leq H(T) \end{cases} \tag{9.21}$$

for any exponent pair (p,q) such that $p > 0$. Raising (9.19) to powers 6 and $2(1 + 2p + 2q)/p$ respectively we obtain

$$(9.22)\qquad R_2 \ll T^{\varepsilon} \sum_{r\le R_2, |\zeta| \ge H(T)} Y^{3-6\sigma} |\zeta(1/2 + it_r)|^6 +$$

$$+ T^{\varepsilon} \sum_{r\le R_2, |\zeta| < H(T)} |\zeta(1/2 + it_r)|^{2(1+2p+2q)/p} Y^{(1+2p+2q)(1-2\sigma)/p} \ll$$

$$\ll T^{1+\varepsilon} Y^{3-6\sigma} + T^{(p+q+\varepsilon)/p} Y^{(1-2\sigma)(1+2p+2q)/p}.$$

Having thus prepared the ground for zero-density estimates we shall proceed to specific results, with the remark that the estimation of R_1 is in general more difficult than the estimation of R_2, for which good bounds (9.20) and (9.22) exist. Several techniques for bounding R_1 will be presented, but it turns out that for $1/2 < \sigma \le 3/4$ the mean value theorem for Dirichlet polynomials (Theorem 5.3) is the best available tool, while for $\sigma \ge 3/4$ the best results are obtained via the Halász-Montgomery inequality (1.35) or (1.36), which offers a considerable flexibility of approach.

§3. The Ingham-Huxley estimates

For the rest of this chapter we shall be proving bounds for $A(\sigma)$ of the type (9.2). The aim of this section is to prove

THEOREM 9.1.

$$(9.23)\qquad A(\sigma) \le 3/(2-\sigma), \qquad 1/2 \le \sigma \le 3/4,$$

$$(9.24)\qquad A(\sigma) \le 3/(3\sigma-1), \qquad 3/4 \le \sigma \le 1,$$

$$(9.25)\qquad A(\sigma) \le 12/5, \qquad 1/2 \le \sigma \le 1.$$

The estimate (9.25) is a simple consequence of (9.23) (due to A.E. Ingham [2]) and (9.24) (due to M.N. Huxley [2]), since for $1/2 \le \sigma \le 3/4$ the function $3/(2-\sigma)$ is increasing, while for $3/4 \le \sigma \le 1$ the function $3/(3\sigma - 1)$ is decreasing and their common value at $\sigma = 3/4$ is $12/5$. The point of (9.25) is that it is the best known estimate of the type $A(\sigma) \le C$ (C an absolute constant) valid for the whole range $1/2 \le \sigma \le 1$, and estimates of this sort are often needed in applications.

Proof of Theorem 9.1. To obtain (9.23) we use (9.17), (9.18) and the mean value theorem for Dirichlet polynomials in the form (5.14). This gives

$$R_1 \ll T^{\varepsilon}\sum_r \Big|\sum_{M<n\le 2M} b(n)n^{-\sigma-i\gamma_r}\Big|^2 \ll T^{\varepsilon}(T+M)M^{1-2\sigma}$$

$$\ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{4(1-2\sigma)/3}),$$

where $\sum_r$ denotes summation over representative zeros $\rho = \beta + i\gamma_r$ of R_1 which satisfy (9.18). Using (9.20) with $M(4) = 1$ it follows from (9.12) that

$$N(\sigma,T) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{4(1-2\sigma)/3} + 1) \ll T^{3(1-\sigma)/(2-\sigma)+\varepsilon}$$

for $Y = T^{3/(8-2\sigma)}$. It is perhaps surprising that (9.23) has been never improved for more than forty years since Ingham's work [2] (except when σ is very close to 1/2), and the main reason for this seems to be that $M(A) = 1$ is still known to hold only for $A \le 4$.

The main difficulty in estimating R_1 in general is the presence of the coefficients $b(n)$, which are non-monotonic and therefore cannot be removed by partial summation techniques such as (1.17) or (1.18). An obvious way to remove the $b(n)$'s is the use of the Halász-Montgomery inequalities, and for the proof of (9.24) we shall use (1.35) with $\xi = \{\xi_n\}_{n=1}^{\infty}$ and $\xi_n = b(n)n^{-\sigma}$ for $M < n \le 2M$ and zero otherwise, $\varphi_r = \{\varphi_{r,n}\}_{n=1}^{\infty}$ and $\varphi_{r,n} = n^{-it_r}$ for $M < n \le 2M$ and zero otherwise, where we have denoted the ordinates of representative zeros of R_1 by $t_1,\dots,t_R$. Then from (9.18) and (1.35) we infer that

$$R_1^2 \ll T^{\varepsilon}R_1M^{2-2\sigma} + T^{\varepsilon}M^{1-2\sigma}\sum_{r\ne s\le R}\Big|\sum_{M<n\le 2M} n^{-it_r+it_s}\Big|.$$

The effect of this procedure is that the inner sum above is an exponential sum to which the techniques of Chapter 2 are applicable. Indeed we estimate this sum as

$$\ll M|t_r - t_s|^{-1} + T^{1/2}$$

by the exponent pair $(p,q) = (1/2,1/2)$ if $|t_r - t_s| \gg M$ and if this is not satisfied then by Lemma 2.5 and Lemma 2.1. Therefore we have

$$R_1^2 \ll T^{\varepsilon} R_1 M^{2-2\sigma} + T^{\varepsilon} M^{2-2\sigma} \sum_{r \neq s \leq R} |t_r - t_s|^{-1} + T^{\varepsilon} R_1^2 T^{1/2} M^{1-2\sigma}.$$

But in view of $|t_r - t_s| \geq \log^4 T$ we have

$$\sum_{r \neq s \leq R} |t_r - t_s|^{-1} \ll R\log T,$$

and therefore we obtain

$$R_1 \ll T^{\varepsilon} M^{2-2\sigma} \tag{9.26}$$

if $T \ll M^{4\sigma-2}$. Thus we divide T into subintervals of length at most $T_o = M^{4\sigma-2}$ so that if R_o denotes the number of representative zeros of R_1 in each of these intervals then (9.26) holds with R_o in place of R_1. Using (9.17) we have then

$$R_1 \ll R_o(1 + T/T_o) \ll T^{\varepsilon}(M^{2-2\sigma} + TM^{4-6\sigma}) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{(16-24\sigma)/3}).$$

From (9.20) with $M(12) \leq 2$ we have finally

$$N(\sigma,T) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{(16-24\sigma)/3} + T^2Y^{6-12\sigma}) \ll T^{3(1-\sigma)/(3\sigma-1)+\varepsilon}$$

with $Y = T^{3/(12\sigma-4)}$, which completes the proof of Theorem 9.1.

§4. Estimates for σ near unity

In this section a different approach to the estimation of R_1 will be presented, again via the Halász-Montgomery inequality (1.35). The method is based on the use of higher power moment estimates of Chapter 7 and gives good bounds for $A(\sigma)$ when σ is close to unity. Only when σ is quite close to unity the bounds connected with zero-free regions for the zeta-function furnish a sharper result than ours, which is

THEOREM 9.2.

$$A(\sigma) \leq 4/(2\sigma + 1), \qquad 17/18 \leq \sigma \leq 1, \tag{9.27}$$

$$A(\sigma) \leq 24/(30\sigma - 11), \qquad 155/174 = 0.8908\ldots \leq \sigma \leq 17/18. \tag{9.28}$$

Proof of Theorem 9.2. As in the proof of (9.24) we shall utilize (1.35), but the choice of ξ and φ_r will be different. We shall take $\xi = \{\xi_n\}_{n=1}^{\infty}$, $\xi_n = b(n)(e^{-n/2M} - e^{-n/M})^{-1/2}n^{-\sigma}$ for $M < n \leq 2M$ and zero otherwise, and $\varphi_r = \{\varphi_{r,n}\}_{n=1}^{\infty}$ with $\varphi_{r,n} = (e^{-n/2M} - e^{-n/M})^{1/2}n^{-it_r}$, $n = 1,2,\ldots$. Writing as before R for the number of representative zeros of R_1 and $t_1,\ldots,t_R$ for their respective ordinates, we obtain from (1.35) and (9.18)

$$(9.29)\qquad R_1^2 \ll T^{\varepsilon}(M^{2-2\delta}R_1 + M^{1-2\delta}\sum_{r\neq s\leq R}|H(it_r - it_s)|),$$

where for t real we have from (4.60) with h = k = 1

$$(9.30)\qquad H(it) = \sum_{n=1}^{\infty}(e^{-n/2M} - e^{-n/M})n^{-it} = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty}\zeta(w+it)((2M)^w-M^w)\Gamma(w)\,dw,$$

since $1 \ll e^{-n/2M} - e^{-n/M} \ll 1$ for $M < n \leq 2M$ and $\|\xi\|^2 \ll T^{\varepsilon}M^{1-2\delta}$, $H(0) \ll M$. Moving the line of integration in (9.30) to $\mathrm{Re}\,w = 1/2$ we encounter a simple pole at $w = 1 - it$ with residue $\ll Me^{-|t|}$ by (1.32), so that

$$(9.31)\qquad H(it) = (2\pi i)^{-1}\int_{1/2-i\infty}^{1/2+i\infty}\zeta(w + it)((2M)^w - M^w)\Gamma(w)\,dw + O(Me^{-|t|}).$$

Also in view of (1.32) the integral for $|\mathrm{Im}\,w| \geq \log^2 T$ in (9.31) is $o(1)$ for $M \ll T^c$, which gives

$$(9.32)\qquad \sum_{r\neq s\leq R}|H(it_r-it_s)| \ll M\sum_{r\neq s\leq R}e^{-|t_r-t_s|} + o(R^2) + M^{1/2}\int_{-\log^2 T}^{\log^2 T}\sum_{r\neq s\leq R}|\zeta(1/2+it_r-it_s+iv)|\,dv.$$

The first sum on the right-hand side of (9.32) is $\ll R$, since by hypothesis the t_r's are at least $\log^4 T$ apart, and for the second sum we fix each s and set $\tau_r = t_r - t_s + v$. Then $|\tau_r| \leq 3T$ for $r = 1,\ldots,R$ and $|\tau_{r_1} - \tau_{r_2}| \geq \log^4 T$ for $r_1 \neq r_2$. We use Hölder's inequality and (9.21) with $(p,q) = (2/7,4/7)$ to obtain

$$\sum_{r\leq R}|\zeta(1/2 + i\tau_r)| \leq R^{5/6}(\sum_{r\leq R,|\zeta|\geq H(3T)}|\zeta(1/2 + i\tau_r)|^6)^{1/6} +$$

$$(9.33)\qquad R^{(2+3p+4q)/(2+4p+4q)}(\sum_{r\leq R,|\zeta|<H(3T)}|\zeta(1/2 + i\tau_r)|^{(2+4p+4q)/p})^{p/(2+4p+4q)}$$

$$\ll T^{\varepsilon}(R^{5/6}T^{1/6} + R^{18/19}T^{3/19}).$$

Inserting (9.33) into (9.32) and using then (9.29) we obtain

$$(9.34) \qquad R_1 \ll T^{\varepsilon}(M^{2-2\sigma} + TM^{9-12\sigma} + T^3M^{19(3-4\sigma)/2}),$$

and to bound R_2 we use (9.22) also with $(p,q) = (2/7,4/7)$, so that

$$(9.35) \qquad R_2 \ll T^{\varepsilon}(TY^{3-6\sigma} + T^3Y^{19(1/2-\sigma)}).$$

We now use first (9.34) to estimate the number of points R_o lying in an interval of length not exceeding $T_o = M^{(72\sigma-53)/6}$. Then $R_o \ll T^{\varepsilon}M^{2-2\sigma}$ for $\sigma \le 11/12$ and

$$(9.36) \qquad R_1 \ll R_o(1 + T/T_o) \ll T^{\varepsilon}(M^{2-2\sigma} + TM^{(65-84\sigma)/6}) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{2(65-84\sigma)/9})$$

for $\sigma \ge 65/84$. With $Y = T^{6/(30\sigma-11)}$ it follows from (9.35) and (9.36) that

$$N(\sigma,T) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{2(65-84\sigma)/9} + TY^{3-6\sigma} + T^3Y^{19(1/2-\sigma)}) \ll$$

$$\ll T^{24(1-\sigma)/(30\sigma-11)+\varepsilon}$$

for $155/174 \le \sigma \le 11/12$. For $\sigma \ge 11/12$ we repeat the procedure choosing this time $T_o = M^{10\sigma-7}$ in (9.34) to obtain $R_o \ll T^{\varepsilon}M^{2-2\sigma}$ and

$$(9.37) \quad R_1 \ll R_o(1 + T/T_o) \ll T^{\varepsilon}(M^{2-2\sigma} + TM^{9-12\sigma}) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{12-16\sigma}) \ll$$

$$\ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{3-6\sigma}).$$

Choosing $Y = T^{1/(2\sigma+1)}$ for $\sigma \ge 17/18$ and $Y = T^{6/(30\sigma-11)}$ for $\frac{11}{12} \le \sigma \le \frac{17}{18}$ respectively we complete the proof on comparing (9.35) and (9.37), and using

$$N(\sigma,T) \ll T^{\varepsilon}(R_1 + R_2 + 1).$$

§5. Reflection principle estimates

All good known zero-density estimates for $\sigma > 3/4$ (but not when σ is very close to unity) involve the use of inequalities of the Halász-Montgomery type, and here we shall follow M. Jutila [2] and derive some estimates which follow from the reflection principle inequality (4.67) with $s = it_r$ (where t_r denotes ordinates of representative zeros of R_1 satisfying (9.18)) and $Y = M$. Similarly to (9.29) we obtain this time

$$(9.38)\qquad R_1^2 \ll T^{\varepsilon}\Big(R_1M^{2-2\sigma} + M^{1-2\sigma}\sum_{r\neq s\le R}|K(it_r - it_s)|\Big),$$

where R is the number of representative zeros of R_1, and where with $h = \log^2 T$ and $t \neq 0$ real we have

$$(9.39)\qquad K(it) = \sum_{n=1}^{\infty}\big(e^{-(n/2M)^h} - e^{-(n/M)^h}\big)n^{-it} \ll 1 + M^{1/2}\int_{-\ell^2}^{\ell^2}\Big|\sum_{n\le 4T/M} n^{-1/2+it+iv}\Big|dv$$

by (4.67). Therefore from (9.38) and (9.39) we infer that

$$(9.40)\qquad R_1^2 \ll T^{\varepsilon}\Big(R_1M^{2-2\sigma} + M^{1-2\sigma}R_1^2 + M^{(3-4\sigma)/2}\int_{-\ell^2}^{\ell^2}\sum_{r\neq s\le R}\Big|\sum_{n\le 4T/M} n^{-1/2+it_r-it_s+iv}\Big|dv.$$

As we are interested in the range $\sigma \ge 3/4$ the term $M^{1-2\sigma}R_1^2$ may be discarded in (9.40), and on applying Hölder's inequality with $k \ge 1$ an integer we have

$$R_1 \ll T^{\varepsilon}M^{2-2\sigma} + T^{\varepsilon}M^{k(3-4\sigma)/2}\max_{|v|\le h^2}\Big(\sum_{r\neq s\le R}\Big|\sum_{n\le 4T/M} n^{-1/2+it_r-it_s+iv}\Big|^{2k}\Big)^{1/2} \ll$$

(9.41)

$$T^{\varepsilon}\Big(M^{2-2\sigma} + M^{k(3-4\sigma)/2}\max_{|v|\le h^2}\ \max_{N\le(4T/M)^k}\Big(\sum_{r\neq s\le R}\Big|\sum_{N<n\le 2N} f(n)n^{-1/2+it_r-it_s+iv}\Big|^2\Big)^{1/2}\Big),$$

where $f(n) \ll n^{\varepsilon}$ and f is independent of t_r and t_s. The point of this approach is that now the coefficients may be removed from the last double sum by appealing to the following simple

<u>Lemma 9.1</u>. Let $a_1,\ldots,a_N$ be complex numbers such that $|a_1| \le A,\ldots,$ $|a_N| \le A$ and let $M \ge N$. Then for any fixed $0 \le \sigma \le 1$

$$(9.42)\qquad \sum_{r,s\le R}\Big|\sum_{n\le N} a_n n^{-\sigma-it_r+it_s}\Big|^2 \le A^2\sum_{r,s\le R}\Big|\sum_{n\le M} n^{-\sigma-it_r+it_s}\Big|^2.$$

<u>Proof of Lemma 9.1</u>. The left-hand side of (9.42) is

$$\sum_{m,n\le N} a_m\bar{a}_n(mn)^{-\sigma}\sum_{r,s\le R}(m/n)^{-it_r+it_s} = \sum_{m,n\le N} a_m\bar{a}_n(mn)^{-\sigma}\Big|\sum_{r\le R}(m/n)^{it_r}\Big|^2 \le$$

$$A^2\sum_{m,n\le M}(mn)^{-\sigma}\Big|\sum_{r\le R}(m/n)^{it_r}\Big|^2 = A^2\sum_{r,s\le R}\Big|\sum_{n\le M} n^{-\sigma-it_r+it_s}\Big|^2.$$

Therefore applying Lemma 9.1 to the last sum in (9.41) with $a_n = f(n)n^{\pm iv}$,

$A = T^{\varepsilon}$, we obtain

$$(9.43)\qquad R_1 \ll T^{\varepsilon}M^{2-2\sigma} + T^{\varepsilon}M^{k(3-4\sigma)/2} \max_{N\le(4T/M)^k} (S(N))^{1/2},$$

where we define

$$(9.44)\qquad S(N) = \sum_{r,s\le R} \Big|\sum_{N<n\le 2N} n^{-1/2-it_r+it_s}\Big|^2, \quad |t_r| \le T,\ |t_r-t_s| \ge \log^4 T \text{ for } r \ne s \le R,$$

and $R \le R_1$ is the representative set of zeros of R_1. Therefore $S(N)$ may be called a "double zeta sum", since it is similar to a (sum of) Dirichlet polynomials approximating $|\zeta(1/2 + it_r - it_s)|$ by the approximate functional equation (4.12). The estimation of $S(N)$ represents the main step in obtaining density estimates from (9.43). The results that will eventually follow then from (9.43) will provide good bounds for $A(\sigma)$ in the range $3/4 < \sigma < 1$, when σ is not close to $3/4$ or 1, and in particular we shall show that $A(\sigma) \le 2$ ("density hypothesis") holds for $\sigma \ge 11/14 = 0.78571...$. In the next section we shall deal with the sums $S(N)$, while in §7 we shall obtain zero-density results from (9.43) and estimates of $S(N)$.

§6. Double zeta sums

There are several ways to treat the double zeta sums $S(N)$ defined by (9.44). First of all note that the terms with $r = s$ in (9.44) contribute $\ll RN$, and if the terms with $r \ne s$ were small one would expect

$$(9.45)\qquad S(N) \ll T^{\varepsilon}(RN + R^2)$$

to hold. This bound is very strong and is, however, certainly well beyond reach at present for all N. Although no restriction on N (with respect to T) has been made in the definition of $S(N)$, one may safely suppose that $N \le T$, since for $N > T$ the sharp bound (9.45) does hold. This is not hard to see since using Lemma 9.1 for $N > T$

$$S(N) \ll RN + N^{-1}\sum_{r\ne s\le R}\Big|\sum_{N<n\le 2N} n^{-it_r+it_s}\Big|^2 \ll$$

$$RN + R^2 + N^{-1}\sum_{r\ne s\le R} N^2|t_r - t_s|^{-2} \ll RN + R^2,$$

where in view of $N > T \ge \frac{1}{2}|t_r - t_s|$ we were able to use Lemma 2.5 and then Lemma 2.1.

Thus in what follows we may always assume that $N \le T$.

Next with $c_n = e^{-n/(2N)} - e^{-n/N}$ we have $1 \ll c_n \ll 1$ for $N < n \le 2N$ and $c_n > 0$ for all $n \ge 1$. Observe that (9.42) remains true with $|a_n| \le Ab_n$ and the inner sum on the right-hand side of (9.42) replaced by $\sum_{n \le M} b_n n^{-\sigma - it_r + it_s}$, so that with $M = \infty$ we obtain

$$(9.46) \qquad S(N) \ll \sum_{r,s \le R} \Big| \sum_{n=1}^{\infty} c_n n^{-1/2 - it_r + it_s} \Big|^2 = S^*(N).$$

The contribution of the terms $r = s$ to $S^*(N)$ is $\ll RN$. To estimate the contribution of the remaining terms note that for t real such that $|t| \ge \log^2 T$ we have

$$(9.47) \qquad \sum_{n=1}^{\infty} c_n n^{-1/2-it} = (2\pi i)^{-1} \int_{1-i\infty}^{1+i\infty} \zeta(w + 1/2 + it)\Gamma(w)((2N)^w - N^w)dw =$$

$$(2\pi i)^{-1} \int_{-i\infty}^{i\infty} \zeta(w + 1/2 + it)\Gamma(w)((2N)^w - N^w)dw + o(1),$$

since the integrand is regular at $w = 0$ (because of the zero of $(2N)^w - N^w$) and the residue at the pole $w = 1/2 - it$ being $o(1)$ by Stirling's formula (1.32)). Likewise the last integral in (9.47) may be broken at $|\mathrm{Im}\, w| = \log^2 T$ with an error $\ll 1$, and we have

$$S^*(N) \ll RN + R^2 + \max_{|v| \le \log^2 T} \sum_{r \ne s \le R} |\zeta(1/2 + it_r - it_s + iv)|^2,$$

so that (9.46) gives

$$(9.48) \qquad S(N) \ll RN + R^2 + \max_{|v| \le \log^2 T} \sum_{r \ne s \le R} |\zeta(1/2 + it_r - it_s + iv)|^2.$$

Here we shall use Hölder's inequality and (9.21) as in the proof of (9.22) to deduce (by fixing each s and summing over r similarly as in (9.33))

$$\sum_{r \ne s \le R} |\zeta(1/2 + it_r - it_s + iv)|^2 \ll \Big(\sum_{|\zeta| \ge H(3T)} |\zeta|^6\Big)^{1/3} R^{4/3} +$$

$$+ \Big(\sum_{|\zeta| < H(3T)} |\zeta|^{(2+4p+4q)/p}\Big)^{p/(1+2p+2q)} R^{(2+2p+4q)/(1+2p+2q)} \ll$$

$$R^{5/3} T^{1/3+\varepsilon} + R^{(2+3p+4q)/(1+2p+2q)} T^{(p+q+\varepsilon)/(1+2p+2q)}.$$

Inserting this bound in (9.48) we obtain

Lemma 9.2. For any exponent pair (p,q) with $p > 0$

(9.49) $$S(N) \ll RN + R^{5/3}T^{1/3+\varepsilon} + R^{(2+3p+4q)/(1+2p+2q)}T^{(p+q+\varepsilon)/(1+2p+2q)}.$$

The particular choice $(p,q) = (1/2,1/2)$ gives immediately

(9.50) $$S(N) \ll RN + R^{11/6}T^{1/3+\varepsilon}.$$

As was remarked already at the beginning of this section, the terms $r = s$ in $S(N)$ make a contribution of order RN, and it seems natural to expect that $S(N)$ is in some sense an increasing function of N (for T fixed). A useful result in this direction will be proved now, which is

Lemma 9.3. For $U \geq N\log T$

(9.51) $$S(N) \ll T^{\varepsilon}S(U).$$

Proof of Lemma 9.3. Let us define for a fixed $K > 1$

$$g_{r,s}(n) = n^{-1/2-it_r+it_s}, \quad S(N,K) = \sum_{r,s\leq R}\Big|\sum_{N<n\leq KN} g_{r,s}(n)\Big|^2$$

with $|t_r| \leq T$, $|t_r - t_s| \geq \log^4 T$ for $r \neq s \leq R$, so that in this notation $S(N) = S(N,2)$. Consider

$$H(e) = \sum_{r,s\leq R}\Big|\sum_{N<n\leq KN} g_{r,s}(n)\Big|^2\Big|\sum_{M<m\leq 3M/2} e_m g_{r,s}(m)\Big|^2$$

$$= \sum_{r,s\leq R}\Big|\sum_{MN<k\leq 3KMN/2} a_k g_{r,s}(k)\Big|^2,$$

where

$$a_k = \sum_{k=mn,\,M<m\leq 3M/2,\,N<n\leq KN} e_m \ll d(k) \ll T^{\varepsilon},$$

if we suppose that $M \ll T^c$, and the components e_m of the vector e are each ± 1. By summing $H(e)$ over $2^{[M/2]}$ possible vectors e and using Lemma 9.1 we obtain

$$\sum_e H(e) \leq 2^{[M/2]}T^{\varepsilon}S(MN, 3K/2).$$

On the other hand

$$\sum_e H(e) = \sum_{r,s\leq R}\Big|\sum_n\Big|^2 \sum_{M<m_1,m_2\leq 3M/2} g_{r,s}(m_1)\overline{g_{r,s}(m_2)}\sum_e e_{m_1}e_{m_2} =$$

$$= \quad 2^{[M/2]} \sum_{r,s\le R} \Big|\sum_n\Big|^2 \sum_{M<m\le 3M/2} m^{-1} \quad \gg \quad 2^{[M/2]} S(N,K),$$

so that

$$S(N,K) \quad \ll \quad T^{\varepsilon} S(MN, 3K/2). \tag{9.52}$$

Here we used the relation

$$\sum_e e_{m_1} e_{m_2} \quad = \quad \begin{cases} 2^{[M/2]} & \text{if } m_1 = m_2, \\ 0 & \text{if } m_1 \ne m_2, \end{cases}$$

since if $m_1 \ne m_2$, then $2^{[M/2]-1}$ summands are +1 and the other $2^{[M/2]-1}$ are -1, cancelling each other. To obtain (9.51) from (9.52) we use first the Cauchy-Schwarz inequality and write

$$S(N) \quad \ll \quad \sum_{j=0}^{2} S(N_j, K), \; K = 2^{1/3}, \; N_j = NK^j, \tag{9.53}$$

and then apply (9.52) with $M = M_j = 1 + [U/N_j]$, so that $U \le M_jN_j$ and for U/N sufficiently large we have $3KM_jN_j/2 \le 2U$. Since $U \ge N\log T$ by hypothesis we have that U/N is large and therefore (9.52) and (9.53) give by the use of Lemma 9.1

$$S(N) \quad \ll \quad T^{\varepsilon} \sum_{j=0}^{2} S(M_jN_j, 3K/2) \quad \ll \quad T^{\varepsilon} S(U).$$

<u>Lemma 9.4</u>.

$$S(N) \quad \ll \quad T^{\varepsilon}(RN \; + \; R^2 \; + \; S(T\log^2 T/N)). \tag{9.54}$$

<u>Proof of Lemma 9.4</u>. As already noted, the result is non-trivial only for $N \le T$. Letting $h = \log^2 T$ we have by the proof of Lemma 9.1 (with $M = \infty$)

$$S(N) \quad \ll \quad N^{-1} \sum_{r,s\le R} \Big|\sum_{n=1}^{\infty} \big(e^{-(n/2N)^h} - e^{-(n/N)^h}\big) n^{-it_r+it_s}\Big|^2,$$

since the exponential term is positive for all $n \ge 1$ and it is $\asymp 1$ for $N < n \le 2N$. Here we use once again the reflection principle estimate (4.67) and Lemma 9.1 to obtain

$$S(N) \quad \ll \quad RN \; + \; R^2 \; + \; T^{\varepsilon} \sum_{r,s\le R} \Big|\sum_{n\le 4T/N} n^{-1/2-it_r+it_s}\Big|^2 \quad \ll$$

$$RN + R^2 + T^{\varepsilon} \sum_{j=0}^{O(\log T)} S(2^{1-j}T/N) \quad \ll \quad RN \; + \; R^2 \; + \; T^{\varepsilon} S(T\log^2 T/N),$$

where in the last step Lemma 9.3 was used. This proves Lemma 9.4.

The preceding lemmas enable us to deduce another explicit estimate for $S(N)$, which is for some ranges of R and N sharper than the bound provided by Lemma 9.2. This is

Lemma 9.5.

$$S(N) \ll T^{\varepsilon}(RN + R^2 + R^{5/4}T^{1/2}), \tag{9.55}$$

and for $N \geq T^{2/3}\log^4 T$ the term $R^{5/4}T^{1/2}$ may be omitted in (9.55).

Proof of Lemma 9.5. By using Lemma 9.1 and Lemma 9.3 we have with the aid of the Cauchy-Schwarz inequality

$$S(N) \leq R\Big(\sum_{r,s\leq R}\Big|\sum_{N<n\leq 2N} n^{-1/2-it_r+it_s}\Big|^4\Big)^{1/2} \ll R(S(2N^2\log T))^{1/2}T^{\varepsilon}.$$

Then using Lemma 9.4 and Lemma 9.3

$$S(N) \ll T^{\varepsilon}RN + T^{\varepsilon}R^2 + T^{\varepsilon}R(S(2T^2N^{-2}\log^5 T))^{1/2} \ll T^{\varepsilon}(RN + R^2 + RS^{1/2}(N))$$

for $N \geq T^{2/3}\log^4 T$, and simplifying we obtain at once (9.55) without $R^{5/4}T^{1/2}$. Therefore the range $N \geq T$ for which the bound (9.45) holds may be extended to $N \geq T^{2/3}\log^4 T$, and any further improvement would be very interesting.

To prove (9.55) we proceed similarly using (9.54) and setting

$$U = \max(N\log T,\ R^{1/4}T^{1/2}\log T).$$

Then

$$S(N) \ll T^{\varepsilon}S(U) \ll T^{\varepsilon}(RU + R^2 + S(T\log^2 T/U)) \ll$$

$$\ll T^{\varepsilon}(RU + R^2 + R(S(T^2U^{-2}\log^5 T))^{1/2}),$$

and using (9.50) we have

$$S(N) \ll T^{\varepsilon}(RU + R^2 + R^{3/2}TU^{-1} + R^{23/12}T^{1/6}) \ll$$

$$\ll T^{\varepsilon}(RN + R^{5/4}T^{1/2} + R^{23/12}T^{1/6}).$$

Repeating the same procedure but using the above estimate in place of (9.50) we obtain

$$S(N) \ll T^{\varepsilon}(RN + R^{47/24}T^{1/12} + R^{5/4}T^{1/2} + R^{13/8}T^{1/4}).$$

But it is easily seen that $R^{13/8}T^{1/4} \leq R^{5/4}T^{1/2} + R^2$, so that the last term in the above estimate may be discarded, and repeating the procedure k times we have

$$S(N) \ll T^{\varepsilon}(RN + R^{5/4}T^{1/2} + R^{2-(6\cdot 2^k)^{-1}}T^{(3\cdot 2^k)^{-1}}),$$

so that (9.55) follows on taking k sufficiently large.

§7. Zero-density theorems for $3/4 < \sigma < 1$.

We now have at disposal good estimates for S(N) furnished by Lemmas of §6, and we shall use (9.43),(9.49),(9.55) to obtain for $k \geq 2$ an integer

$$R_1 \ll T^{\varepsilon}\Big\{M^{2-2\sigma} + M^{k(3-4\sigma)/2}R_1 + M^{k(1-2\sigma)}T^{k/2}R_1^{1/2} +$$

$$M^{k(3-4\sigma)/2}\min(R_1^{5/8}T^{1/4}, R_1^{5/6}T^{1/6} + R_1^{(2+3p+4q)/(2+4p+4q)}T^{(p+q)/(2+4p+4q)})\Big\}.$$

In view of $\sigma > 3/4$ the term $M^{k(3-4\sigma)/2}R_1$ may be omitted and after simplifying the above estimate we obtain

$$(9.56) \qquad R_1 \ll T^{\varepsilon}\Big\{M^{2-2\sigma} + M^{k(2-4\sigma)}T^k +$$

$$+ \min(T^{2/3}M^{4k(3-4\sigma)/3}, TM^{3k(3-4\sigma)} + T^{(p+q)/p}M^{\frac{k}{p}(1+2p+2q)(3-4\sigma)})\Big\}.$$

To make the first two terms on the right-hand side of (9.56) equal we choose

$$(9.57) \qquad T = T_0 = M^{((4k-2)\sigma+2-2k)/k}.$$

With this choice of T_0 the remaining terms do not exceed $M^{2-2\sigma}$ for

$$(9.58) \qquad \sigma \geq \min\left\{\frac{6k^2-5k+2}{8k^2-7k+2}, \max\left(\frac{9k^2-4k+2}{12k^2-6k+2}, \frac{3k^2(1+2p+2q)-(4p+2q)k+2p+2q}{4k^2(1+2p+2q)-(6p+4q)k+2p+2q}\right)\right\}.$$

Thus we obtain, provided that (9.58) holds,

$$R_1 \ll T^{\varepsilon}M^{2-2\sigma}(1 + T/T_0) \ll T^{\varepsilon}(M^{2-2\sigma} + TM^{2-2\sigma}M^{((2k-2+(2-4k)\sigma)/k)}) \ll$$

$$T^{\varepsilon}(Y^{4-4\sigma} + TY^{4(4k-2-(6k-2)\sigma)/3k}),$$

since $4k-2-(6k-2)\sigma \leq 0$ for $\sigma \geq 3/4$ and $k \geq 2$. Using (9.20) with $M(12) \leq 2$ we have

$$(9.59) \qquad N(\sigma,T) \ll T^{\varepsilon}(Y^{4-4\sigma} + TY^{4(4k-2-(6k-2)\sigma)/3k} + T^2Y^{6-12\sigma}) \ll$$

$$\ll T^{3k(1-\sigma)/((3k-2)\sigma+2-k)+\varepsilon}$$

for $Y = T^{3k/((12k-8)\sigma+8-4k)}$.

Before we proceed to estimates arising from specific values of k it may be noted that letting $k \to \infty$ all the expressions in (9.58) tend to 3/4 and (9.59) gives then $A(\sigma) \leq 3/(3\sigma - 1)$ when k is very large, which is just Huxley's estimate (9.24). The functions appearing in (9.58) are decreasing functions of k, while the function that appears in (9.59) in the estimate for $N(\sigma,T)$ is an increasing function of k, so that there is no simple k which will furnish the sharpest bound obtainable by this method in the whole range $\sigma \geq \sigma_0 > 3/4$. Taking first k = 2 we see that $A(\sigma) \leq 3/(2\sigma)$ holds for

$$(9.60) \qquad \sigma \geq \min(4/5, \max(\frac{15}{19}, \frac{6 + 9p + 11q}{8 + 11p + 13q})),$$

and the choice $(p,q) = (97/251, 132/251)$ gives $A(\sigma) \leq 3/(2\sigma)$ for $\sigma \geq \frac{3831}{4791} = 0.799624$.
For k = 3 we have $A(\sigma) \leq 9/(7\sigma - 1)$ for

$$(9.61) \qquad \sigma \geq \min(41/53, \max(\frac{71}{92}, \frac{27 + 55p + 50q}{36 + 56p + 62q})),$$

and therefore $A(\sigma) \leq 9/(7\sigma - 1)$ holds for $\sigma \geq 41/53 = 0.773584\ldots$. Since $9/(7\sigma - 1) \leq 2$ for $\sigma \geq 11/14$ we obtain also

$$(9.62) \qquad A(\sigma) \leq 2 \quad \text{for } \sigma \geq 11/14 = 0.785714\ldots,$$

which is the best known range for which the density hypothesis holds. Finally we shall also consider the case k = 4, when looking at the first expression on the right-hand side of (9.58) we see that $A(\sigma) \leq 6/(5\sigma - 1)$ holds for $\sigma \geq 13/17 = 0.764705\ldots$. The above estimates may be collected together to give

THEOREM 9.3.

$$(9.63) \qquad A(\sigma) \leq 3/(2\sigma) \quad \text{for } 3831/4791 = 0.799624\ldots \leq \sigma \leq 1,$$

$$(9.64) \qquad A(\sigma) \leq 2 \quad \text{for } 11/14 = 0.785714\ldots \leq \sigma \leq 1,$$

$$(9.65) \qquad A(\sigma) \leq 9/(7\sigma - 1) \quad \text{for } 41/53 = 0.773584\ldots \leq \sigma \leq 1,$$

$$(9.66) \qquad A(\sigma) \leq 6/(5\sigma - 1) \quad \text{for } 13/17 = 0.764705\ldots \leq \sigma \leq 1.$$

§8. Zero-density estimates for σ close to 3/4

Although all estimates of the type (9.59) improve (9.24) for a fixed k, none can be made yet to hold in the whole range $\sigma \geq 3/4$. Therefore it seems of interest to try to find an estimate which will improve (9.24) for $\sigma \geq 3/4$. This may be done, and the result is

THEOREM 9.4.

$$(9.67)\qquad A(\sigma) \le 3/(7\sigma - 4) \qquad \text{for } 3/4 \le \sigma \le 10/13,$$

$$(9.68)\qquad A(\sigma) \le 9/(8\sigma - 2) \qquad \text{for } 10/13 \le \sigma \le 1 .$$

Note that both (9.67) and (9.24) give $A(3/4) \le 3/5$, but Theorem 9.4 improves (9.24) in the whole range $\sigma > 3/4$. Neverthelesss (9.66) is still sharper, so that the importance of (9.67) lies in the range $3/4 < \sigma \le 13/17$. The main idea of proof is the use of a double zeta-function sum (different from (9.44)) at the line $\sigma = 3/4$. For a fixed θ satisfying $1/2 < \theta < 1$ let us define

$$(9.69)\qquad S_1(\theta) = \sum_{r,s\le R} |\zeta(\theta + it_r - it_s + iv')|^2,$$

where the real numbers $t_1,\dots,t_R$ satisfy $|t_r| \le T$, $|t_r - t_s| \ge 2\log^4 T$ for $r \ne s \le R$ and v' is defined by

$$(9.70)\qquad |\zeta(\theta + it_r - it_s + iv')| = \max_{-\log^2 T\le v\le \log^2 T} |\zeta(\theta + it_r - it_s + iv)| .$$

Furthermore we define $S_1(1/2)$ analogously as $S_1(\theta)$, only for technical reasons in the definition of v' the maximum of v is to be taken over the interval $[-2\log^2 T, 2\log^2 T]$. The proof of Theorem 9.4 will require a good bound for $S_1(3/4)$, which is furnished by

Lemma 9.6.

$$(9.71)\qquad S_1(3/4) \ll T^{\varepsilon}(R^2 + R^{11/8}T^{1/4}).$$

Proof of Lemma 9.6. The most important step in the proof of (9.71) consists in showing that

$$(9.72)\qquad S_1(3/4) \ll T^{\varepsilon}R^2 + T^{\varepsilon}R^{3/4}(S_1(1/2))^{1/2}.$$

To obtain (9.72) we may start from (9.4) with $s = 3/4 + it_r - it_s + iv'$ $(r \ne s)$, $1 \ll Y \ll T^C$, and move the line of integration to $\mathrm{Re}\, w = -1/4$. We encounter a pole at $w = 1 - s$ with residue $o(1)$ in view of (1.32) and a pole at $w = 0$ with residue $\zeta(s)$. Therefore

$$(9.73)\qquad \zeta^2(s) \ll 1 + \Big|\sum_{n\le Y} e^{-n/Y} n^{-s}\Big|^2 + Y^{-1/2}\int_{-\log^2 T}^{\log^2 T} e^{-|y|}\,|\zeta(1/2+it_r-it_s+iv'+iy)|^2 dy,$$

and summing over $r,s \le R$ it follows by the Cauchy-Schwarz inequality

$$(9.74)\qquad S_1(3/4) \ll T^{\varepsilon}R^2 + \sum_{r,s\le R}\Big|\sum_{n\le Y} e^{-n/Y} n^{-3/4-it_r+it_s-iv'}\Big|^2 + Y^{-1/2}S_1(1/2) \ll$$

$$T^{\varepsilon}R^2 + R\Big(\sum_{r,s\le R}\Big|\sum_{n\le Y^2} c(n)n^{-3/4-it_r+it_s-iv'}\Big|^2\Big)^{1/2} + Y^{-1/2}S_1(1/2),$$

where $c(n) \ll n^{\varepsilon}$. To estimate the sum under the square root we use Lemma 9.1 and obtain, similarly as in (9.44),

$$\sum_{r,s\le R}\Big|\sum_{n\le Y^2}\Big|^2 \ll \log T\cdot \max_{M\le Y^2}\sum_{r,s\le R}\Big|\sum_{M<n\le 2M}\Big|^2 \ll T^{\varepsilon}\max_{M\le Y^2} M^{-3/2}\sum_{r,s\le R}|H(it_r - it_s)|^2,$$

where for t real

$$H(it) = \sum_{n=1}^{\infty}(e^{-n/2M} - e^{-n/M})n^{-it} = (2\pi i)^{-1}\int_{2-i\infty}^{2+i\infty}\zeta(w+it)((2M)^w-M^w)\Gamma(w)\,dw,$$

and therefore trivially $H(0) \ll M$. For $t = t_r - t_s \neq 0$ we move the line of integration in the above integral to $\operatorname{Re} w = 3/4$, encountering a pole at $w = 1 - it_r + it_s$ with residue $o(1)$ by (1.32), and we obtain

$$\sum_{r,s\le R}\Big|\sum_{n\le Y^2} c(n)n^{-3/4-it_r+it_s}\Big|^2 \ll T^{\varepsilon}\max_{M\le Y^2} M^{-3/2}(RM^2 + R^2 + M^{3/2}S_1(3/4)) \ll$$

$$\ll T^{\varepsilon}(RY + R^2 + S_1(3/4)).$$

Inserting this estimate in (9.74) and simplifying, we obtain with $Y = R^{-3/2}S_1(1/2)$

$$S_1(3/4) \ll T^{\varepsilon}R^2 + T^{\varepsilon}R^{3/2}Y^{1/2} + Y^{-1/2}S_1(1/2) \ll T^{\varepsilon}(R^2 + R^{3/4}(S_1(1/2))^{1/2}),$$

which is precisely (9.72).

To obtain finally (9.71) from (9.72) we need an adequate bound for $S_1(1/2)$. From the reflection principle equation (4.66) we have with $s = 1/2 + it_r - it_s + iv'$ $(r \neq s)$, $h = \log^2 T$, $\alpha = 1/2 - \varepsilon$, $M = 4T/Y$, $k=1$

$$\zeta(s) \ll 1 + \Big|\sum_{n\le Y} e^{-(n/Y)^h} n^{-s}\Big| + T^{\varepsilon}\int_{-h^2}^{h^2}\Big|\sum_{n\le 4T/Y} n^{s-1-\varepsilon+iy}\Big|dy. \tag{9.75}$$

To make the lengths of the sums in (9.75) equal we choose $Y = 2T^{1/2}$. Squaring, summing over $r,s \le R$, using Lemma 9.1 and Lemma 9.3 it follows that

$$S_1(1/2) \ll T^{\varepsilon}(R^2 + S(2T^{1/2}\log T)), \tag{9.76}$$

where $S(N)$ is defined by (9.44). If we use (9.55) to bound $S(2T^{1/2}\log T)$, then (9.71) follows at once from (9.72) and (9.76).

<u>Proof of Theorem 9.4</u>. It remains yet to prove Theorem 9.4 with the aid

of Lemma 9.6. We use (9.29), but now in (9.30) we move the line of integration to $\mathrm{Re}\,w = 1/4$ and employ the functional equation (1.1) for the zeta-function. Instead of (9.32) we obtain now with the aid of the Cauchy-Schwarz inequality

$$\sum_{r\neq s\leq R} |H(it_r - it_s)| \ll M \sum_{r\neq s\leq R} e^{-|t_r - t_s|} + R^2 + (MT)^{1/4} \int_{-\log^2 T}^{\log^2 T} \sum_{r\neq s\leq R} |\zeta(\tfrac{3}{4} + it_r - it_s + iv)|dv$$

$$\ll RM + R^2 + T^{\varepsilon+1/4} M^{1/4} R (S_1(3/4))^{1/2}.$$

Therefore if we use this estimate and Lemma 9.6 in (9.29), it follows after some simplification

$$(9.77) \qquad R_1 \ll T^{\varepsilon} M^{2-2\sigma} + R_1 T^{\varepsilon+1/4} M^{(5-8\sigma)/4} + T^{\varepsilon+6/5} M^{(20-32\sigma)/5}.$$

For R_0 points lying in an interval of length $T = T_0 = M^{8\sigma-5-\varepsilon}$ we have

$$R_0 \ll T^{\varepsilon}(M^{2-2\sigma} + T_0^{6/5} M^{(20-32\sigma)/5}) \ll T^{\varepsilon} M^{2-2\sigma}$$

for $\sigma \leq 10/13$, and therefore for $3/4 \leq \sigma \leq 10/13$ with $Y = T^{3/(28\sigma-16)}$ we obtain

$$R_1 \ll R_0(1 + T/T_0) \ll T^{\varepsilon} M^{2-2\sigma}(1 + T/T_0) \ll$$

$$T^{\varepsilon}(Y^{4-4\sigma} + TY^{(28-40\sigma)/3}) \ll T^{3(1-\sigma)/(7\sigma-4)+\varepsilon}.$$

Using (9.20) with $M(4) = 1$ it is seen that for $3/4 \leq \sigma \leq 11/14$ one has $TY^{2-4\sigma} \leq TY^{(28-40\sigma)/3}$, and (9.67) follows. Analogously we obtain (9.68) if in (9.77) we choose $T_0 = M^{(11\sigma-5)/3}$.

N O T E S

A classical application of zero-density estimates consists in bounding $p_{n+1} - p_n$ from above, where p_n is the n-th prime number. The main tool is the well-known formula of E. Landau (K. Chandrasekharan [1], Chapter 5)

$$(9.78) \qquad \psi(x) = \sum_{n\leq x} \Lambda(n) = \sum_{p^m \leq x} \log p = x - \sum_{|\gamma|\leq T} x^{\rho}\rho^{-1} + O(xT^{-1}\log^2 x), \ T \leq x,$$

where $|\gamma| \leq T$ denotes summation over all zeros $\rho = \beta + i\gamma$ of the zeta-function counted by $N(\sigma,T)$. If using (9.78) one can prove

$$(9.79) \qquad \psi(x + h) - \psi(x) = (1 + o(1))h, \ h = x^{\theta}, \ 1/2 \leq \theta < 1,$$

then taking $x = p_n$ one easily deduces from (9.79)

(9.80) $$p_{n+1} - p_n \ll p_n^{\theta}.$$

However if one forms $\psi(x+h) - \psi(x)$ with the aid of (9.78), then one encounters the sum

(9.81) $$\sum_{|\gamma|\le T} \frac{(x+h)^{\rho} - x^{\rho}}{\rho} = \sum_{|\gamma|\le T} \int_x^{x+h} z^{\rho-1}dz \ll h \sum_{|\gamma|\le T} x^{\beta-1} \ll \log x \cdot \max_{\sigma} x^{\sigma-1} N(\sigma,T),$$

where the maximum is over the interval

(9.82) $$0 < \sigma \le 1 - C\log^{-2/3}x\cdot(\log\log x)^{-1/3},\ C > 0,$$

since for $\beta > 1 - C\log^{-2/3}|\gamma|(\log\log|\gamma|)^{-1/3}$ there are no zeros of the zeta-function, as shown by A. Walfisz [2] with the use of I.M. Vinogradov's method of estimating exponential sums (for a somewhat weaker zero-free region one may consult E.C. Titchmarsh [8] and K. Chandrasekharan [1], while for an elementary approach to the zero-free region (9.82) the reader is referred to articles of Y. Motohashi [1], [2]). Thus (9.81) shows how inequalities of the type (9.80) are connected with zero-density estimates, and if $A(\sigma) \le D$ for $1/2 \le \sigma \le 1$ and some absolute $D \ge 2$, then using (9.81) and (9.82) it follows that (9.80) holds with any $\theta > 1 - D^{-1}$. Thus (9.25) gives $\theta = 7/12 + \varepsilon$, the density hypothesis gives $\theta = 1/2 + \varepsilon$, while under the Riemann hypothesis nothing more than $p_{n+1} - p_n \ll p_n^{1/2}\log p_n$ (proved by H. Cramér [2]) is known, though Cramér himself already conjectured an incomparably sharper bound, namely $p_{n+1} - p_n \ll \log^2 p_n$, which seems altogether hopeless for today's methods. For a long time the only approach to (9.80) was the one just outlined, but recently H. Iwaniec and M. Jutila [1] have successfully combined sieve and analytic techniques to obtain $\theta = 13/23$ in (9.80), and this approach was further elaborated by D.R. Heath-Brown and H. Iwaniec [1], where $\theta = 11/20 + \varepsilon$ was obtained. In a preprint of the Math. Institute of the Hungarian Academy of Sciences J. Pintz showed that the crucial Lemma 2 of Heath-Brown - Iwaniec [1] can be improved, which leads to $\theta = 17/31 + \varepsilon$. Pintz also announces the value $\theta = 23/42 + \varepsilon = 0.547619\ldots + \varepsilon$ which will appear in a joint paper with Iwaniec.

For a comprehensive study of works concerning zero-density estimates the reader is referred to Chapter 12 of H.L. Montgomery [2], where a general form of the zero-detection method is given (to include zero-density estimates for L functions), as well as several sharp bounds for $N(\sigma,T)$ when σ is close to 1, including

(9.83) $$N(\sigma,T) \ll T^{167(1-\sigma)^{3/2}}\log^{17}T, \quad 1/2 \le \sigma \le 1,$$

which is connected with the zero-free region (9.82).

Concerning sharp bounds when σ is close to 1/2 one should mention the bound

(9.84) $$N(\sigma,T) \ll T^{1-\frac{1}{4}(\sigma-1/2)}\log T, \quad 1/2 \le \sigma \le 1$$

of A. Selberg [1] . Recently M. Jutila [3] has improved (9.84) by replacing 1/4 by $1 - \delta$ for any fixed $0 < \delta < 1$.

The useful procedure of estimating R_1 by its representatives for which (9.16) and (9.18) hold has been introduced by M. Jutila [1], while (as mentioned in Chapter 7) the technique of dividing T into subintervals of length T_0 and then multiplying the resulting estimate by $1 + T/T_0$ is due to M.N. Huxley [1].

A.E. Ingham [2] proved $N(\sigma,T) \ll T^{3(1-\sigma)/(2-\sigma)}\log^5 T$ by a method different from the one presented in §2, and which seems to be more complicated. Ingham's result, with a slightly weaker log-factor, can be obtained from the zero-detection method of §2 when one does not choose $X = T^{\varepsilon}$ but $X = T$, $Y = T^{3/(4-2\sigma)}$ and uses again the mean value theorem for Dirichlet polynomials and a discrete form of the fourth power moment. The details may be found in Chapter 12 of Montgomery [2] or in Chapter 23 of Huxley [1]. A similar discussion holds in connection with Huxley's bound (9.24), as he proved in [2] $N(\sigma,T) \ll T^{3(1-\sigma)/(3\sigma-1)}\log^{44}T$.

As stated in the proof of Ingham's estimate (9.22), the main reason why this bound withstands improvement for more than forty years is the lack of $M(A) = 1$ for $A > 4$. This explains also the seemingly mysterious choice $r = 2$ in (9.16). Namely if in (9.16) we choose $r = 3$ ($r \ge 4$ can be analyzed analogously), then following the proof of Theorem 9.1 we have

$$R_1 \ll T^{\varepsilon}M^{1-2\sigma}(T + M) \ll T^{\varepsilon}(Y^{6-6\sigma} + TY^{9(1-2\sigma)/4}), \quad 1/2 \le \sigma \le 3/4,$$

$$R_1 \ll T^{\varepsilon}(M^{2-2\sigma} + TM^{4-6\sigma}) \ll T^{\varepsilon}(Y^{6-6\sigma} + TY^{9(2-3\sigma)/2}), \quad 3/4 \le \sigma \le 1.$$

If we now use $R_2 \ll T^{1+\varepsilon}Y^{2-4\sigma}$ or $R_2 \ll T^{2+\varepsilon}Y^{6-12\sigma}$ (coming from $M(4) = 1$ and $M(12) \le 2$ respectively) we see that we get poorer estimates for $N(\sigma,T)$ than the ones furnished by Theorem 9.1. However if $M(6) = 1$ were known to hold, then from (9.20) and the above estimates we would obtain

$$N(\sigma,T) \ll T^{\epsilon}(Y^{6-6\sigma} + TY^{9(1-2\sigma)/4} + TY^{3-6\sigma}) \ll T^{8(1-\sigma)/(5-2\sigma)+\epsilon}, \quad 1/2 \le \sigma \le 3/4,$$

with $Y = T^{4/(15-6\sigma)}$ and

$$N(\sigma,T) \ll T^{\epsilon}(Y^{6-6\sigma} + TY^{9(2-3\sigma)/2} + TY^{3-6\sigma}) \ll T^{4(1-\sigma)/(5\sigma-2)+\epsilon}, \quad 3/4 \le \sigma \le 1$$

with $Y = T^{2/(15\sigma-6)}$. This would improve Theorem 9.1, giving also $A(\sigma) \le 16/7$ for the whole range $1/2 \le \sigma \le 1$. This is one more reason for the importance of the sixth power moment estimate $M(6) = 1$.

The simplest way to obtain (9.22) is to note that from (9.19) one has $\zeta(1/2 + it_r) \gg T^{-\epsilon}Y^{\sigma-1/2}$ for $r = 1,\dots,R_2$, so that (7.8) can be applied directly with $V = T^{-\epsilon}Y^{\sigma-1/2}$, giving (9.22).

Theorem 9.2 is due to the author [2], and improves on $A(\sigma) \le 4/(4\sigma-1)$, $25/18 \le \sigma \le 1$, which was obtained by D.R. Heath-Brown [4]. For

$$\sigma \ge 1 - \left(\frac{4}{3 \cdot 167}\right)^2 = 0.99993625\ldots$$

the estimate (9.83) supersedes (9.27). The exponent pair $(p,q) = (2/7,4/7)$ that was used in the proof of Theorem 9.2 gives by no means the best result, but other exponent pairs would lead to more complicated formulas and slight improvements only.

If the Lindelöf hypothesis that $\zeta(1/2 + it) \ll t^{\epsilon}$ is true, then trivially for $Y \gg T^{\epsilon_1}$ one has $R_2 \ll T^{\epsilon}$ for $\sigma > 1/2$, and by (9.32)

$$\sum_{r,s \le R} |H(it_r - it_s)| \ll RM + R^2M^{1/2}T^{\epsilon},$$

so that by (9.29) one has for $\sigma > 3/4$

$$R_1 \ll T^{\epsilon}M^{2-2\sigma} \ll T^{\epsilon}Y^{4-4\sigma}.$$

Choosing $Y = T^{\epsilon}$ one obtains

$$N(\sigma,T) \ll T^{\epsilon}, \text{ for } \sigma \ge 3/4 + \delta,$$

where $\epsilon = \epsilon(\delta)$ may be made arbitrarily small for any $\delta > 0$, which is a result of G. Halász and P. Turán [1]. For a nice survey of Turán's conditional and unconditional results concerning density estimates the reader is referred to P. Turán [1].

§5 is based on M. Jutila's paper [2], which amomg other things contains the bound $\sigma \ge 11/14$ for which density $A(\sigma) \le 2$ holds, and this is still the best known result of this type. The history of bounds for density is also mentioned in Jutila [2] and is as follows. Let a be such a constant for which the density hypothesis $A(\delta) \leq 2$ holds for $\delta \geq a$. Then H.L. Montgomery [2] proved $a \leq 9/10$, M.N. Huxley [2] $a \leq 5/6$, K. Ramachandra [2] $a \leq 21/26$, F. Forti and C. Viola [1] $a \leq 0.8059\ldots$, M.N. Huxley [3] $a \leq 4/5$ and some intermediate results, M. Jutila [1] $a \leq 43/54$.

Lemma 9.2 is due to the author, while the remaining lemmas of §6 are due to D.R. Heath-Brown [6]. An alternative proof of Lemma 9.5 may be given with the aid of (9.48). If we use (9.75), Lemma 9.3 and Lemma 9.4 we obtain

$$S_1(1/2) \ll T^{\varepsilon}(S(2Y\log T) + S(2TY^{-1}\log T)) \ll T^{\varepsilon}(RY + R^2 + S(2TY^{-1}\log^3 T)).$$

But now by the Cauchy-Schwarz inequality we have

$$S(2TY^{-1}\log^3 T) \ll T^{\varepsilon}R(S(T^2Y^{-2}\log^8 T))^{1/2} \ll$$

$$\ll T^{\varepsilon}(R^{3/2}TY^{-1} + R^2 + S_1^{1/2}(1/2)R),$$

where we used again Lemma 9.1, Lemma 9.3 and (9.48), and $S_1(1/2)$ is the double zeta-sum that appears in (9.48). From the last two bounds we have

$$S_1(1/2) \ll T^{\varepsilon}(RY + R^{3/2}TY^{-1} + R^2) \ll T^{\varepsilon}(R^{5/4}T^{1/2} + R^2)$$

with $Y = R^{1/4}T^{1/2}$, and Lemma 9.5 follows then at once from (9.48).

Ranges given for various estimates by Theorem 9.3 are the best ones known. (9.63) was proved by M.N. Huxley to hold for $\delta \geq 37/42$ in [3], and by the author [1] for $\delta \geq 4/5$, while the present range was given by the author in [2]. (9.65) was proved by D.R. Heath-Brown [4] for $\delta \geq 11/14$ and by the author [1] for $\delta \geq 74/95$. A proof of (9.66) for $\delta \geq 67/87$ is also given in the author's paper [1], where there are also given zero-density bounds coming from $c(\theta)$ (defined by (6.51)).

Lemma 9.6 and Theorem 9.4 are due to the author [4]. An argument similar to the one used in the proof of Lemma 9.6 gives

$$S(\theta) \ll T^{\varepsilon}R^2 + T^{\varepsilon}R^{3\theta-3/2}(S(1/2))^{2-2\theta},$$

where $S(\theta)$ is defined by (9.69), and the above estimate for $\theta = 3/4$ reduces to (9.72).

Aleksandar Ivić

TOPICS IN RECENT ZETA-FUNCTION THEORY

C H A P T E R 10

D I V I S O R   P R O B L E M S

CHAPTER 10

DIVISOR PROBLEMS

§1. Introduction

In this chapter we shall investigate various problems involving $\Delta_k(x)$, the error term in the asymptotic formula for $\sum_{n\le x} d_k(n)$, where for $k \ge 2$ fixed $d_k(n)$ is the number of ways n can be written as a product of k factors. For $\mathrm{Re}\, s > 1$ we have therefore $\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n)n^{-s}$, which shows at once the intrinsic connection between $\Delta_k(x)$ and the zeta-function, and thus it is natural to expect that properties of $\Delta_k(x)$ and $\zeta^k(s)$ are closely connected. This is even more apparent as by the inversion formula (1.8) one has

$$\sum_{n\le x}{}' d_k(n) = (2\pi i)^{-1} \int_{2-i\infty}^{2+i\infty} \zeta^k(s)x^s s^{-1} ds. \tag{10.1}$$

Moving the line of integration to some $1/2 < c < 1$ (but sufficiently close to 1) it is seen that the integrand in (10.1) has only a pole of order k at $s = 1$, and so by the residue theorem

$$\sum_{n\le x}{}' d_k(n) = xP_{k-1}(\log x) + (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \zeta^k(s)x^s s^{-1} ds, \quad 1/2 < c < 1, \tag{10.2}$$

where $P_{k-1}(t)$ is a polynomial of degree $k - 1$ in t. If we write

$$\Delta_k(x) = \sum_{n\le x} d_k(n) - xP_{k-1}(\log x), \tag{10.3}$$

then coefficients of P_{k-1} may be evaluated by using

$$P_{k-1}(\log x) = \operatorname*{Res}_{s=1} x^{s-1}\zeta^k(s)s^{-1}. \tag{10.4}$$

Namely starting from the Laurent expansion

$$\zeta(s) = 1/(s-1) + \gamma + \sum_{k=1}^{\infty} \gamma_k (s-1)^k, \quad \gamma_k = \frac{(-1)^k}{k!} \lim_{N\to\infty} \Big\{ \sum_{m\le N} \frac{1}{m}\log^k m - \frac{\log^{k+1} N}{k+1} \Big\}, \tag{10.5}$$

one may calculate explicitly the coefficients of $P_{k-1}(t)$ as functions of the γ_k's ($\gamma = \gamma_0$ = Euler's constant = 0.5772157...), and for instance we have

$$(10.6)\qquad P_1(t) = t + (2\gamma - 1),$$

$$(10.7)\qquad P_2(t) = \tfrac{1}{2}t^2 + (3\gamma-1)t + (3\gamma^2 - 3\gamma + 3\gamma_1 + 1),$$

$$(10.8)\qquad P_3(t) = \tfrac{1}{6}t^3 + (2\gamma - 1/2)t^2 + (6\gamma^2 - 4\gamma + 4\gamma_1 + 1)t +$$

$$+ (-1 + 4(\gamma - \gamma_1 + \gamma_2) - 6\gamma^2 + 4\gamma^3 + 12\gamma\gamma_1),$$

and in general the coefficients of $P_{k-1}(t)$ may be found by formulas of A.F. Lavrik et al. [1].

This chapter is the direct counterpart of Chapter 12 of E.C. Titchmarsh's book [8], and the aim here is to give an overall improvement of results presented by Titchmarsh. The notation is however the same, and in particular we define α_k and β_k as the infima of numbers a_k and b_k respectively for which

$$(10.9)\qquad \Delta_k(x) \ll x^{a_k+\varepsilon}, \quad \int_1^x \Delta_k^2(y)dy \ll x^{1+2b_k+\varepsilon}.$$

One of our main topics will be the determination of upper bounds for α_k and β_k, but in later sections of this chapter we shall investigate some other related problems involving $\Delta_k(x)$. The chapter ends with a discussion of the circle problem, whose close connection with the divisor problem for k = 2 is exhibited.

§2. Estimates for $\Delta_2(x)$ and $\Delta_3(x)$

The most convenient way of obtaining estimates for $\Delta_2(x)$ and $\Delta_3(x)$ seems to be the use of

$$(10.10)\qquad \Delta_k(x) \ll x^{(k-1)/2k}\Big|\sum_{n\le N} d_k(n)n^{-(k+1)/2k}e(k(nx)^{1/k})\Big| + x^{(k-1+\varepsilon)/k}N^{-1/k} + x^{\varepsilon},$$

as given by (3.23). By writing

$$d_k(n) = \sum_{m_1 m_2 \ldots m_k = n} 1$$

the sum over $n \le N$ in (10.10) is transformed then into a multiple exponential sum, and the best results hitherto seem to be those obtainable by methods of G. Kolesnik [5], [6]. A closer look at Titchmarsh's proof of (10.10) in [8] reveals that for k = 2 one has

(10.11) $$\Delta_2(x) \ll x^{1/4}\Big|\sum_{mn\le N}(mn)^{-3/4}e(2(mn)^{1/2})\Big| + x^{1/2}N^{-1/2}\log^2 x + x^{\varepsilon},$$

since in Titchmarsh's proof one may take $a = \varepsilon$ and $c = 1 + 1/\log x$. Therefore an application of Lemma 6.3 gives similarly as in the proof of Theorem 6.3

(10.12) $$\Delta_2(x) \ll x^{\varepsilon} + \log^2 x(\max_{M\le N}(x^{3/16}M^{173/152-3/4} + x^{5/16}M^{119/152-3/4})+x^{1/2}N^{-1/2})$$

$$\ll x^{\varepsilon} + \log^2 x(x^{3/16}N^{59/152} + x^{5/16}N^{5/152} + x^{1/2}N^{-1/2}) \ll x^{35/108}\log^2 x$$

for $N = x^{19/54}$. Therefore we obtain

THEOREM 10.1.

(10.13) $$\Delta_2(x) \ll x^{35/108}\log^2 x.$$

Here the exponent 35/108 is exactly twice the exponent for the order of $\zeta(1/2 + iT)$ in (6.63). This is no coincidence, since the exponential sums to which both problems reduce are of a very similar nature. More light on the intrinsic connection between $\Delta_2(x) = \Delta(x)$ and $\zeta(1/2 + iT)$ will be shed in the last chapter.

The estimation of $\Delta_3(x)$ is naturally more complicated than the estimation of $\Delta_2(x)$, and is carried out via (10.10) with $k = 3$. The best result yet is

(10.14) $$\Delta_3(x) \ll x^{43/96+\varepsilon}.$$

This is due to G. Kolesnik [5], and the proof is long and complicated and will not be presented here.

§3. Estimates of $\Delta_k(x)$ by power moments of the zeta-function

From the Perron inversion formula (1.10) we have with $c = 1 + \varepsilon$, $T \le x$,

(10.15) $$\sum_{n\le x} d_k(n) = (2\pi i)^{-1}\int_{c-iT}^{c+iT}\zeta^k(s)x^s s^{-1}ds + O(x^{1+\varepsilon}T^{-1}),$$

since the contribution $\frac{1}{2}d_k(x)$ (if x is an integer) counted by $\sum'$ in (1.10) is absorbed in the error term $O(x^{1+\varepsilon}T^{-1})$. For $1/2 \le \sigma < 1$ fixed we deform the path of integration in the above integral to obtain by the residue theorem

$$(10.16)\qquad \Delta_k(x) = \sum_{n\le x} d_k(n) - \operatorname*{Res}_{s=1} \zeta^k(s)x^s s^{-1} = I_1 + I_2 + I_3 + O(x^{1+\varepsilon}T^{-1}),$$

say, where

$$(10.17)\qquad I_1 = (2\pi i)^{-1}\int_{\sigma-iT}^{\sigma+iT} \zeta^k(s)x^s s^{-1}ds \ll x^\sigma + x^\sigma\int_1^T |\zeta(\sigma + it)|^k t^{-1}dt$$

and

$$(10.18)\qquad I_2 + I_3 \ll \int_\sigma^{1+\varepsilon} x^\theta|\zeta(\theta + iT)|^k T^{-1}d\theta \ll x^{1+\varepsilon}T^{-1} + x^\sigma T^{kc(\sigma)-1+\varepsilon},$$

where $c(\theta)$ is defined by (6.51). From (10.17) it is immediately seen that estimates for power moments of the zeta-function lead to estimates of $\Delta_k(x)$. Our result will be the following

THEOREM 10.2. Let α_k be the infimum of numbers a_k such that $\Delta_k(x) \ll x^{a_k+\varepsilon}$ for every $\varepsilon > 0$. Then

$$\alpha_k \le (3k - 4)/4k \qquad \text{for } 4 \le k \le 8,$$
$$\alpha_9 \le 35/54,\quad \alpha_{10} \le 41/60,\quad \alpha_{11} \le 7/10,$$
$$\alpha_k \le (k - 2)/(k + 2) \qquad \text{for } 12 \le k \le 25,$$
$$\alpha_k \le (k - 1)/(k + 4) \qquad \text{for } 26 \le k \le 50,$$
$$\alpha_k \le (31k - 98)/32k \qquad \text{for } 51 \le k \le 57,$$
$$\alpha_k \le (7k - 34)/7k \qquad \text{for } k \ge 58.$$

Proof of Theorem 10.2. The proof is based on estimates of $m(\sigma)$, as furnished by Theorem 7.3. For a fixed integer k we choose σ in such a way that $m(\sigma) = k$, where for $m(\sigma)$ we take the estimates which are given by Theorem 7.3. From bounds for $c(\theta)$ given in Chapter 6 it is seen that $m(\sigma) \le 1/c(\sigma)$, so that taking $T = x^{1-\sigma}$ in (10.17) and (10.18) we obtain

$$\Delta_k(x) \ll x^{\sigma+\varepsilon}.$$

In this fashion estimates for $9 \le k \le 11$ given by Theorem 10.2 follow at once, and for $4 \le k \le 8$ we use $m(\sigma) = 4/(3 - 4\sigma)$ $(1/2 \le \sigma \le 5/8)$, so that $k = 4/(3 - 4\sigma)$ gives $\sigma = (3k - 4)/4k$. For $4 \le k \le 8$ this value of σ satisfies $1/2 \le \sigma \le 5/8$ and $\alpha_k \le (3k - 4)/4k$ follows for $4 \le k \le 8$. Next we take $\sigma = 5/7$ in (10.17) and (10.18). With $m(5/7) \ge 12$, $c(5/7) \le 1/14$ we have

$$I_1 \ll x^{5/7} + x^{5/7}\int_1^T |\zeta(5/7 + it)|^{12}t^{-1}|\zeta(5/7 + it)|^{k-12}dt \ll$$

$$\ll x^{5/7}T^{(k-12+\varepsilon)/14}$$

for $k \geq 12$ and therefore

$$\Delta_k(x) \ll x^{1+\varepsilon}T^{-1} + x^{5/7}T^{(k-12+\varepsilon)/14} \ll x^{(k-2)/(k+2)+\varepsilon}$$

for $12 \leq k \leq 25$ if $T = x^{4/(k+2)}$.

A similar argument gives $\alpha_k \leq (k-1)/(k+4)$ for $k \geq 26$ by using $m(5/6) > 26$, $c(5/6) \leq 1/30$. Also by Theorem 7.3 we have $m(\sigma) \geq 98/(31-32\sigma) = k$ for $13/15 \leq \sigma = (31k-98)/32k \leq 0.91591\ldots$, which is satisfied for $30 \leq k \leq 57$. By (7.88) we have $m(\sigma) \geq 34/(7-7\sigma) = k$ for $\sigma = (7k-34)/7k \geq 0.91591\ldots$ for $k \geq 57$. On comparing then $(k-1)/(k+4)$ with $(31k-98)/32k$ we obtain the full assertion of Theorem 10.2.

For each particular $k \geq 13$ the bounds of Theorem 10.2 can be slightly improved by a more careful choice of exponent pairs in bounds furnished by (7.65), and taking more care one could also derive bounds of the type $\Delta_k(x) \ll x^{\alpha_k}\log^{D_k}x$ for some $D_k \geq 0$. The bounds of Theorem 10.2 are the sharpest ones known, except when k is very large, when better bounds may be obtained by using the best known zero-free region of the zeta-function, which will be the topic of the next section.

§4. Estimates of $\Delta_k(x)$ when k is very large

We shall end our order estimates of $\Delta_k(x)$ by proving

THEOREM 10.3. There is an absolute $C > 0$ such that for $k \geq k_0$

$$\alpha_k \leq 1 - Ck^{-2/3}. \tag{10.19}$$

This estimate is clearly seen to improve on Theorem 10.2 for $k \geq k_1$, hence for "k very large". The value of the constant C which appears in (10.19) depends on order estimates of the zeta-function near the line $\sigma = 1$ and may be explicitly evaluated with some effort, but it is much more difficult to determine k_0 such that (10.19) holds for $k \geq k_0$. The order result needed for the proof of Theorem 10.3 is contained in

Lemma 10.1. There exists an absolute constant $D > 0$ such that uniformly in $1/2 \leq \sigma \leq 1$ we have for $t \geq t_0$

$$\zeta(\sigma + it) \ll t^{D(1-\sigma)^{3/2}}\log t. \tag{10.20}$$

Proof of Lemma 10.1. The proof of Lemma 10.1 is based on the simple approximate functional equation

$$(10.21)\qquad \zeta(s) = \sum_{n\le x} n^{-s} + x^{1-s}/(s-1) + O(x^{-\sigma}),\ x > t/\pi > t_0,\ 0 < \sigma < 1,$$

proved in Notes of Chapter 7, and the estimate

$$(10.22)\qquad \sum_{N<n\le N_1\le 2N} n^{it} \ll N\exp\left(-B\frac{\log^3 N}{\log^2 t}\right),\ N \le t,$$

where $B > 0$ is some absolute constant. The estimate (10.22) is in fact a consequence of I.M. Vinogradov's well-known method of estimating exponential sums (see A.A. Karacuba [3], Ch. 5). We take $x = t$ in (10.21) and split the sum over $n \le x$ into $O(\log T)$ subsums of the type $\sum_{N<n\le 2N} n^{-s}$. Using partial summation and (10.22) we obtain

$$(10.23)\qquad \sum_{N<n\le 2N} n^{-s} \ll N^{1-\sigma}\exp\left(-B\frac{\log^3 N}{\log^2 t}\right) \ll t^{D(1-\sigma)^{3/2}}$$

with $D = 2/(3\sqrt{3B})$, since

$$N^{1-\sigma}\exp\left(-B\frac{\log^3 N}{\log^2 t}\right) = \exp\left((1-\sigma)\log N - B\frac{\log^3 N}{\log^2 t}\right),$$

and the function $f(x) = Bx^3\log^{-2}t - (1-\sigma)x\ (x > 0)$ attains a minimum at

$$x = \left(\frac{1}{3B}(1-\sigma)\log^2 t\right)^{1/2}.$$

The estimate (10.20) follows at once from (10.23).

Proof of Theorem 10.3. In (10.17) and (10.18) choose $\sigma = 1 - k^{-2/3}$, $T = x^{Ek^{-2/3}}$, where $E > 0$ will be suitably chosen in a moment. Then by (10.20)

$$x^{\sigma}\int_1^T |\zeta(\sigma + it)|^k t^{-1}dt \ll x^{\sigma}\log T \max_{1\le T_1\le T}\int_{T_1}^{2T_1} |\zeta(\sigma + it)|^k t^{-1}dt \ll$$

$$x^{\sigma}T^{Dk(1-\sigma)^{3/2}}\log^{k+1}x \ll x^{1-k^{-2/3}+DEk^{-2/3}}\log^{k+1}x = x^{1-0.5k^{-2/3}}\log^{k+1}x$$

if $E = 1/2D$. From (10.18) we have

$$I_2 + I_3 \ll \max_{\sigma\le\theta\le 1+\varepsilon} x^{\theta}T^{D(1-\theta)^{3/2}-1} \ll x^{1+\varepsilon+Ek^{-2/3}(Dk^{-1}-1)} \ll$$

$$\ll x^{1+\varepsilon-\frac{E}{2}k^{-2/3}}$$

for $k > 2D$, hence Theorem 10.3 follows from the above estimates.

§5. Estimates of β_k

We recall that β_k is the infimum of b_k for which $\int_0^x \Delta_k^2(y)dy \ll x^{1+2b_k+\varepsilon}$ holds for every $\varepsilon > 0$, so that β_k may be thought of as the exponent of the average order of $|\Delta_k(y)|$. The classical elementary results concerning the estimation of β_k are embodied in the following two lemmas which may be found in Chapter 12 of Titchmarsh [8], but which will be given here for the sake of completeness of the exposition.

Lemma 10.2. Let γ_k be the infimum of $\sigma > 0$ for which

$$\int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k}|\sigma + it|^{-2}dt \ll 1.$$

Then $\beta_k = \gamma_k$ and for $\sigma > \beta_k$

$$(2\pi)^{-1}\int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k}|\sigma + it|^{-2}dt = \int_0^{\infty} \Delta_k^2(x)x^{-2\sigma-1}dx. \tag{10.24}$$

Proof of Lemma 10.2. From (10.2) we have

$$\Delta_k(x) = (2\pi i)^{-1} \lim_{T\to\infty}\int_{c-iT}^{c+iT} \zeta^k(s)x^s s^{-1}ds \tag{10.25}$$

for some $c < 1$ and close to 1. Since $\zeta^k(s)s^{-1} \to 0$ uniformly in the strip as $t \to \pm\infty$, it is seen on integrating over the rectangle $c' \pm iT$, $c \pm iT$, $\gamma_k < c' < c < 1$ that (10.25) holds for any $c > \gamma_k$. Replacing x by $1/x$, taking $c > \gamma_k$ and using Parseval's identity (1.5) we have

$$(2\pi)^{-1}\int_{-\infty}^{\infty} |\zeta(c + it)|^{2k}|c+it|^{-2}dt = \int_0^{\infty}\Delta_k^2(1/x)x^{2c-1}dx = \int_0^{\infty}\Delta_k^2(x)x^{-2c-1}dx, \tag{10.26}$$

which gives for $\gamma_k < c < 1$

$$\int_N^{2N} \Delta_k^2(x)x^{-2c-1}dx \ll 1, \quad \int_N^{2N} \Delta_k^2(y)dy \ll N^{2c+1},$$

and therefore

$$\int_0^x \Delta_k^2(y)\,dy = \sum_{j\geq 1}\sum_{N=x2^{-j}} \int_N^{2N} \Delta_k^2(y)\,dy \ll x^{2c+1},$$

hence $\beta_k \leq c$, $\beta_k \leq \gamma_k$.

The other inequality, namely $\beta_k \geq \gamma_k$, may be obtained by observing that from (10.25) and Mellin's formula (1.1) one has

$$(10.27) \qquad \zeta^k(s)s^{-1} = \int_0^\infty \Delta_k(1/x)x^{s-1}dx = \int_0^\infty \Delta_k(x)x^{-s-1}dx.$$

The integral in (10.27) is absolutely and uniformly convergent for $\beta_k < \sigma < 1$, since by the Cauchy-Schwarz inequality

$$\int_N^{2N} |\Delta_k(x)|x^{-\sigma-1}dx \leq (\int_N^{2N} \Delta_k^2(x)dx)^{1/2}(\int_N^{2N} x^{-2\sigma-2}dx)^{1/2} \ll N^{\beta_k-\sigma+\varepsilon},$$

and by adding integrals over various $[N,2N]$ it is seen that the right-hand side of (10.27) is regular for $\beta_k < \sigma < 1$, so that (10.27) holds by analytic continuation in the strip $\beta_k < \sigma < 1$. By the same argument the right-hand side of (10.25) is bounded for $\beta_k < \sigma < 1$, hence (10.25) holds in the same strip, giving $\beta_k \geq \gamma_k$, which combined with $\beta_k \leq \gamma_k$ yields finally $\beta_k = \gamma_k$.

<u>Lemma 10.3</u>. For $k = 2,3,\ldots$

$$\alpha_k \geq \beta_k \geq (k-1)/2k.$$

<u>Proof of Lemma 10.3</u>. The inequality $\alpha_k \geq \beta_k$ is obvious, and for the other inequality we start from

$$T \ll \int_{T/2}^{T} |\zeta(\sigma+it)|^2 dt \leq (\int_{T/2}^{T} |\zeta(\sigma+it)|^{2k}dt)^{1/k}(\int_{T/2}^{T} dt)^{1-1/k},$$

where the lower bound for $1/2 < \sigma < 1$ follows easily by termwise integration of $\zeta(s) = \sum_{n\leq t} n^{-s} + O(t^{-\sigma})$, or directly from (7.99). Therefore we obtain

$$(10.28) \qquad T \ll \int_{T/2}^{T} |\zeta(\sigma+it)|^{2k}dt,$$

where the $\ll$-constant depends only on k. Using the functional equation we have for $0 < \sigma < 1/2$

$$\int_{-\infty}^{\infty} |\zeta(\sigma+it)|^{2k}|\sigma+it|^{-2}dt \geq \int_{T/2}^{T} |\zeta(\sigma+it)|^{2k}|\sigma+it|^{-2}dt \gg$$

$$\gg T^{-2}\int_{T/2}^{T}|\zeta(\sigma + it)|^{2k}dt \gg T^{k(1-2\sigma)-2}\int_{T/2}^{T}|\zeta(1-\sigma-it)|^{2k}dt \gg T^{k(1-2\sigma)-1},$$

where in the last step (10.28) was used. For $\sigma < (k - 1)/2k$ the last expression remains unbounded, giving $\gamma_k \geq (k - 1)/2k$, and the result follows from $\beta_k = \gamma_k$.

<u>Lemma 10.4</u>. For each integer $k \geq 2$ a necessary and sufficient condition that $\beta_k = (k - 1)/2k$ is that $m((k+1)/2k) \geq 2k$, where $m(\sigma)$ is defined by (7.2).

<u>Proof of Lemma 10.4</u>. Suppose first that $m((k+1)/2k) \geq 2k$. Then for $\sigma < (k - 1)/2k$ we have by the functional equation

$$\int_1^T|\zeta(\sigma + it)|^{2k}dt \ll T^{k(1-2\sigma)}\int_1^T|\zeta(1 - \sigma - it)|^{2k}dt \ll T^{k(1-2\sigma)+1+\varepsilon}.$$

Therefore for $(k - 1 - \varepsilon)/2k < \sigma < (k + 1 + \varepsilon)/2k$ by convexity of mean values we have

$$\int_1^T|\zeta(\sigma + it)|^{2k}dt \ll T^{1+\varepsilon+(1/2+1/2k-\sigma)k},$$

and the exponent of T is < 2 for $\sigma > (k - 1 + \varepsilon)/2k$, giving

$$\int_{T/2}^{T}|\zeta(\sigma + it)|^{2k}|\sigma + it|^{-2}dt \ll T^{-\delta}$$

for some $\delta = \delta(\varepsilon) > 0$. Replacing T by $T/2, T/2^2$, etc. it follows that $\gamma_k \geq (k-1)/2k$, and so by Lemma 10.2 $\beta_k \geq (k-1)/2k$ also.

In the other direction, if $\beta_k = (k-1)/2k$, then by (10.24)

$$\int_1^T|\zeta(\sigma + it)|^{2k}dt \ll T^{2+\varepsilon}$$

for $\sigma > (k-1)/2k$, and using convexity of mean values and the functional equation we obtain $m((k+1)/2k) \geq 2k$ by following the argument just given.

The lemmas that were just presented show how the estimation of β_k may be reduced to obtaining sufficiently sharp estimates for $m(\sigma)$. We shall prove the following

<u>THEOREM 10.4.</u> $\beta_k = (k-1)/2k$ for $k = 2,3,4$ and $\beta_5 \leq 119/260 = 0.45769\ldots$, $\beta_6 \leq 1/2$, $\beta_7 \leq 39/70 = 0.55714\ldots$.

<u>Proof of Theorem 10.4</u>. By Theorem **10.2** we have $m(\sigma) \geq 4/(3 - 4\sigma)$ for $1/2 \leq \sigma \leq 5/8$, hence $m(5/8) \geq 8$ and so by Lemma 10.4 we obtain at once that $\beta_k = (k-1)/2k$ for $k = 2,3,4$, which in view of Lemma 10.3 shows that this is best possible. For other values of k the estimate $\beta_k = (k-1)/2k$ seems to be beyond

reach at present, as is also the classical conjecture $\alpha_k = \beta_k = (k-1)/2k$ for $k = 2,3,\dots$.

Consider now the case $k = 5$. By Lemma 10.2 it will suffice to show

$$\int_T^{2T} |\zeta(\sigma + it)|^{10} dt \ll T^{2-\delta}$$

for $\sigma > 119/260$ and any fixed $\delta > 0$. From the estimate $m(41/60) \geq 10$, furnished by Theorem 7.3, and the functional equation for the zeta-function we have for $19/60 \leq \sigma \leq 1/2$

$$\int_T^{2T} |\zeta(\sigma + it)|^{10} dt \ll T^{(207-260\sigma)/44+\varepsilon},$$

where we used convexity and the estimate $M(10) \leq 7/4$ of Theorem 7.2. Since $(207 - 260\sigma)/44 < 2$ for $\sigma > 119/260$ we obtain $\beta_5 \leq 119/260$ as asserted. Similarly from $M(12) \leq 2$ it follows at once that $\beta_6 \leq 1/2$, while for β_7 we use $M(14) \leq \frac{62}{27}$ (Theorem 7.2) and $m(3/4) > 14$ (Theorem 7.3). This gives by convexity

$$\int_T^{2T} |\zeta(\sigma + it)|^{14} dt \ll T^{(132-140\sigma)/27+\varepsilon}$$

for $1/2 \leq \sigma \leq 3/4$, and $(132 - 140\sigma)/27 < 2$ for $\sigma > 39/70$, proving the last part of the theorem. Other values of β_k for $k \geq 8$ may be calculated analogously, but the present form of estimates for $m(\sigma)$ and $M(A)$ would render a general formula for β_k $(k \geq 8)$ too complicated, and for this reason only estimates for small values of k are explicitly stated here.

§6. Mean square estimates of $\Delta_k(x)$

By definition estimates of β_k are in fact mean square estimates of $\Delta_k(x)$. However owing to the importance of the integrals in question it seems appropriate to investigate them more closely. In particular it would be highly desirable to obtain asymptotic formulas for $\int_1^T \Delta_k^2(x)dx$, and we begin the discussion of this problem by proving

THEOREM 10.5.

$$\int_1^T \Delta_2^2(x)dx = (6\pi^2)^{-1}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}T^{3/2} + O(T^{5/4+\varepsilon}). \tag{10.29}$$

Proof of Theorem 10.5. The value of the constant $\sum_{n=1}^{\infty} d^2(n)n^{-3/2}$ is

$\zeta^4(3/2)/\zeta(3) = 14.8316\ldots$. It will be sufficient to prove the corresponding estimate for $\int_T^{2T}$ and then to replace T by $T/2, T/2^2$, etc. and to add up all the results. We start from the truncated Voronoï formula (3.17), where we take $N = T$. Integrating term by term we obtain

$$\int_T^{2T} \Delta_2^2(x)\,dx = (2\pi^2)^{-1}\int_T^{2T} x^{1/2} \sum_{m,n\le T} d(m)d(n)(mn)^{-3/4}\cos(4\pi\sqrt{mx}-\pi/4)\cos(4\pi\sqrt{nx}-\pi/4)\,dx +$$

(10.30)

$$+ \quad O\Big(T^{1/4+\varepsilon}\int_T^{2T}\Big|\sum_{n\le T} d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)\Big|\,dx\Big) \quad + \quad O(T^{1+\varepsilon}).$$

In the first sum in (10.30) we distinguish the cases $m = n$ and $m \ne n$. The terms with $m = n$ contribute

$$(2\pi^2)^{-1}\sum_{n\le T}\int_T^{2T} d^2(n)n^{-3/2}x^{1/2}\cos^2(4\pi\sqrt{nx}-\pi/4)\,dx =$$

(10.31)
$$(4\pi^2)^{-1}\sum_{n\le T} d^2(n)n^{-3/2}\int_T^{2T} x^{1/2}(1+\cos(8\pi\sqrt{nx}-\pi/2))\,dx =$$

$$(6\pi^2)^{-1}((2T)^{3/2} - T^{3/2})\sum_{n=1}^{\infty} d^2(n)n^{-3/2} \quad + \quad O(T\log^3 T).$$

In (10.31) we have used partial summation and (5.24) to obtain

$$\sum_{n>T} d^2(n)n^{-3/2} \ll T^{-1/2}\log^3 T,$$

and we have used (2.3) to estimate

$$\sum_{n\le T} d^2(n)n^{-3/2}\int_T^{2T} x^{1/2}\cos(8\pi\sqrt{nx}-\pi/2)\,dx \ll T\sum_{n\le T} d^2(n)n^{-2} \ll T.$$

In view of $2\cos X\cos Y = \cos(X+Y) + \cos(X-Y)$ it is seen that the terms in (10.30) for which $m \ne n$ are a multiple of

(10.32)
$$\sum_{m\ne n\le T} d(m)d(n)(mn)^{-3/4}\int_T^{2T} x^{1/2}\cos(4\pi\sqrt{mx}-4\pi\sqrt{nx})\,dx \quad +$$

$$\sum_{m\ne n\le T} d(m)d(n)(mn)^{-3/4}\int_T^{2T} x^{1/2}\sin(4\pi\sqrt{mx}+4\pi\sqrt{nx})\,dx = S_1 + S_2,$$

say. Estimating the integrals in S_2 by (2.3) we have

(10.33)
$$S_2 \ll T\sum_{m<n\le T} d(m)d(n)(mn)^{-3/4}(m^{1/2}+n^{1/2})^{-1} \ll T\log^3 T.$$

Analogously we obtain

$$(10.34)\qquad S_1 \ll T\sum_{n<m\le T} d(m)d(n)(mn)^{-3/4}(m^{1/2} - n^{1/2})^{-1} = T\Big(\sum_{n\le m/2} + \sum_{n>m/2}\Big) =$$

$$= T(S_1' + S_1''),$$

say. We have

$$S_1' \ll \sum_{m\le T} d(m)m^{-1/4}\sum_{n\le m/2} d(n)n^{-3/4}(m-n)^{-1} \ll$$

$$\ll \sum_{m\le T} d(m)m^{-5/4}m^{1/4}\log T \ll \log^3 T,$$

$$S_1'' \ll \sum_{m\le T} d(m)m^{-1}\sum_{m/2<n<m} d(n)(m-n)^{-1} \ll T^{\varepsilon}\sum_{m\le T} d(m)m^{-1} \ll T^{\varepsilon}.$$

Therefore the first sum in (10.30) is by preceding estimates equal to

$$(6\pi^2)^{-1}((2T)^{3/2} - T^{3/2})\sum_{n=1}^{\infty} d^2(n)n^{-3/2} + O(T^{1+\varepsilon}).$$

The first O-term is estimated in (10.30) by the Cauchy-Schwarz inequality as

$$\ll T^{3/4+\varepsilon}\Big(\int_T^{2T}\Big|\sum_{n\le T} d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \pi/4)\Big|^2 dx\Big)^{1/2} \ll T^{5/4+\varepsilon}$$

when we square out the modulus under the integral sign and treat the terms $m = n$ and $m \ne n$ similarly as before.

This remark ends the proof of Theorem 10.5, but it should be observed that the error term given in Theorem 10.5 is by no means the best possible one. Analyzing more carefully the proof it may be seen that T^{ε} in the error term in (10.30) may be replaced by a suitable log-power, but this would be still much weaker than the following result of K.-C. Tong [2] :

$$(10.35)\qquad \int_1^T \Delta_2^2(x)dx = (6\pi^2)^{-1}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}T^{3/2} + O(T\log^5 T).$$

The proof of this formula is beyond the scope of the method used for Theorem 10.5, and requires subtle averaging techniques involving certain exponential integrals. It seems also natural to ask what is the best possible O-result that may be obtained in (10.35). In this direction we shall prove the following

THEOREM 10.6. The asymptotic formula

$$(10.36)\qquad \int_1^T \Delta_2^2(x)\,dx = (6\pi^2)^{-1}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}T^{3/2} + O(T^{3/4-\delta})$$

cannot hold for any $\delta > 0$.

Proof of Theorem 10.6. From Theorem 10.5 it is seen that there exist arbitrarily large x such that

$$(10.37)\qquad |\Delta_2(x)| = |\Delta(x)| > Cx^{1/4} = G$$

for some suitable $C > 0$, and from the classical work of G.H. Hardy [2] it follows that C may be taken arbitrary. Suppose now that $|t - x| \le Gx^{-\varepsilon}$. Since $d(n) < n^{\varepsilon/3}$ for $n \ge n_0(\varepsilon)$ we have in view of (10.3)

$$(10.38)\qquad |\Delta(t) - \Delta(x)| \le \Big|\sum_{t\le n\le x} d(n)\Big| + \Big|xP_1(\log x) - tP_1(\log t)\Big| \le Gx^{-\varepsilon/2},$$

and then also

$$(10.39)\qquad |\Delta(t)| \ge \Big||\Delta(x)| - |\Delta(t) - \Delta(x)|\Big| \ge G - Gx^{-\varepsilon/2} \ge G/2.$$

Next by the Cauchy-Schwarz inequality and (10.39)

$$(10.40)\qquad G^2x^{-\varepsilon} \le \int_{x-Gx^{-\varepsilon}}^{x+Gx^{-\varepsilon}} |\Delta(t)|\,dt \le (2Gx^{-\varepsilon})^{1/2}\Big(\int_{x-Gx^{-\varepsilon}}^{x+Gx^{-\varepsilon}} \Delta^2(t)\,dt\Big)^{1/2} \le$$

$$\le (2Gx^{-\varepsilon})^{1/2}\Big(\tfrac{3}{2}D\cdot 2Gx^{-\varepsilon}\cdot 2x^{1/2} + \max_{|t-x|\le Gx^{-\varepsilon/2}} |R(t)|\Big)^{1/2},$$

where we have set

$$(10.41)\qquad R(x) = \int_1^x \Delta^2(y)\,dy - Dx^{3/2},\quad D = (6\pi^2)^{-1}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}.$$

If $R(x) \ll x^{3/4-\delta}$ for some $\delta > 0$, then (10.40) yields

$$(10.42)\qquad G^2x^{-\varepsilon} \le (12D)^{1/2}Gx^{1/4-\varepsilon} + O(G^{1/2}x^{-\varepsilon/2}x^{3/8-\delta/2}).$$

Now if in (10.37) we choose $C > (12D)^{1/2}$ and take $\varepsilon = \delta/2$, then for x sufficiently large (10.42) gives a contradiction which proves the theorem, and small improvements may be obtained by using sharper Ω-estimates for $\Delta(x)$ than (10.37).

It seems natural to ask whether the same type of results such as those furnished by Theorem 10.5 and 10.6 hold also for $\Delta_k(x)$ when $k \ge 3$. In considering this problem it ought to be mentioned that K.-C. Tong [2] proved a general result concerning asymptotic formulas for $\int_1^x \Delta_k^2(y)dy$, which seems to be hitherto the sharpest one. If in analogy with (10.41) we define

$$(10.43) \qquad R_k(x) = \int_1^x \Delta_k^2(y)dy - ((4k-2)\pi^2)^{-1}\sum_{n=1}^{\infty} d_k^2(n)n^{-(k+1)/k}x^{(2k-1)/k},$$

then Tong's result may be formulated as

$$(10.44) \qquad R_k(x) \ll \begin{cases} x\log^5 x, & k = 2 \\ x^{c_k+\varepsilon}, \quad c_k = 2 - \dfrac{3 - 4\sigma_k}{2k(1-\sigma_k)-1}, & k \geq 3, \end{cases}$$

where σ_k is the infimum of σ such that for every $\varepsilon > 0$

$$(10.45) \qquad \int_1^T |\zeta(\sigma + it)|^{2k}dt \ll T^{1+\varepsilon},$$

and for (10.44) to hold one should have $\sigma_k \leq (k+1)/2k$.

Suppose now that $k = 3$. Then by Theorem 7.3 we have $m(7/12) \geq 6$, which in the notation of (10.45) implies $\sigma_3 \geq 7/12$, hence from (10.44) we infer

$$(10.46) \qquad \int_1^x \Delta_3^2(y)dy = (10\pi^2)^{-1}\sum_{n=1}^{\infty} d_3^2(n)n^{-4/3}x^{5/3} + O(x^{14/9+\varepsilon}).$$

This is substantially stronger than $\beta_3 = 1/3$ only, as given by Theorem 10.4, but with $k = 3$ (10.44) at present exhausts itself in the sense that for $k > 4$ the best estimates for σ_k obtainable from Theorem 7.3 are not sufficiently sharp to ensure that the condition $\sigma_k \leq (k+1)/2k$ is satisfied, and in the case $k = 4$ we have $\sigma_4 \leq 5/8$ which gives only $R_4(x) \ll x^{7/4+\varepsilon}$, and this is equivalent to $\beta_4 = 3/8$, which was already established by Theorem 10.4. A result analogous to Theorem 10.6 may be obtained in the general case by defining θ_k as the infimum of θ such that for every $\varepsilon > 0$

$$(10.47) \qquad R_k(x) \ll x^{\theta+\varepsilon}, \quad \theta < 2 - 1/k.$$

Arguing as in the case $k = 2$ it follows that $\theta_k < (3k - 3)/2k$ cannot hold, but I think that it is reasonable to conjecture that $\theta_k = (3k - 3)/2k$. This conjecture, however, seems to be very strong as it implies at once the classical conjecture $\alpha_k = (k-1)/2k$ by the following

THEOREM 10.7. Let θ_k be the infimum of numbers θ ($< 2 - 1/k$) such that (10.47) holds for every $\varepsilon > 0$. Then $\alpha_k \le \frac{1}{3}\theta_k$.

Proof of Theorem 10.7. The proof is analogous to the proof of Theorem 10.6. Suppose that $R_k(x) \ll x^{\theta_k+\varepsilon/2}$ for $x \ge x_0$ and some $\varepsilon > 0$ and suppose that for some sufficiently large x and suitable $C > 0$

$$(10.48)\qquad |\Delta_k(x)| > Cx^{\theta_k/3+\varepsilon} = G.$$

Then for $|t - x| \le Gx^{-\varepsilon}$ we have as in (10.38) and (10.39) that

$$|\Delta_k(t)| \ge G/2,$$

and therefore using (10.43) and the Cauchy-Schwarz inequality

$$(10.49)\qquad G^2x^{-\varepsilon} \le (2Gx^{-\varepsilon})^{1/2}\Big(\int_{x-Gx^{-\varepsilon}}^{x+Gx^{-\varepsilon}} \Delta_k^2(t)\,dt\Big)^{1/2} \le$$

$$(2Gx^{-\varepsilon})^{1/2}(E_kGx^{-\varepsilon}x^{(k-1)/k} + O(x^{\theta_k+\varepsilon/2}))^{1/2}$$

for some $E_k > 0$, and we obtain from (10.49)

$$(10.50)\qquad G^2x^{-\varepsilon} \le (2E_k)^{1/2}Gx^{-\varepsilon}x^{(k-1)/2k} + O(G^{1/2}x^{\theta_k/2-\varepsilon/4}).$$

However (10.50) is seen to be impossible for $C > (2E_k)^{1/2}$ in (10.48) because $\theta_k \ge (3k-3)/2k$. Therefore (10.48) cannot hold and we obtain $\alpha_k \le \frac{1}{3}\theta_k$ as asserted.

From (10.35) and (10.46) we have $\theta_2 \le 1$, $\theta_3 \le 14/9$, so that from Theorem 10.7 we deduce $\alpha_2 \le 1/3$, $\alpha_3 \le 14/27$, which is superseded by (10.13) and (10.14). In fact the estimates for α_2 and α_3 can be deduced directly from estimates of $\int_{T-G}^{T+G} \Delta_k^2(y)\,dy$. Using the method of proof of Theorem 10.5 it is readily seen that for $T^\varepsilon \le G \le T$

$$(10.51)\qquad \int_{T-G}^{T+G} \Delta_2^2(t)\,dt \ll T^\varepsilon(GT^{1/2} + T),$$

$$(10.52)\qquad \int_{T-G}^{T+G} \Delta_3^2(t)\,dt \ll T^\varepsilon(GT^{2/3} + T^{3/2}),$$

which gives $\alpha_2 \le 1/3$, $\alpha_3 \le 1/2$ following the method of proof of Theorem 10.7.

§7. Large values and power moments of $\Delta_k(x)$

In view of Theorem 10.5 and (10.46) it is seen that in mean square $\Delta_2(x)$ and $\Delta_3(x)$ are of the order $x^{1/4}$ and $x^{1/3}$ respectively, which supports the conjecture $\alpha_2 = 1/4$, $\alpha_3 = 1/3$. An interesting problem is to generalize mean square estimates to higher powers, and to consider integrals of the type $\int_1^T |\Delta_k(x)|^A dx$, $A \ge 2$. As the starting point of these investigations one may take (3.23), namely

$$\Delta_k(x) \ll x^{(k-1)/2k}\Big|\sum_{n\le N} d_k(n)n^{-(k+1)/2k}e(k(nx)^{1/k})\Big| + x^{(k-1+\varepsilon)/k}N^{-1/k} + x^{\varepsilon}. \tag{10.53}$$

The sum over $n \le N$ for $k = 2$ is similar in nature to the sum occurring in the investigation of large values of the zeta-function on the critical line in Chapter 7, and the methods that we shall apply to deal with the large values of $\Delta_k(x)$ will be similar. The main obstacle is the presence of the divisor function $d_k(n)$ in (10.53), which will be eliminated by the use of the Halász-Montgomery inequality. This will lead to a large values estimate for $\Delta_k(x)$, and then (similarly as was done for higher power moments of the zeta-function in Chapter 7) we shall estimate $\int_1^T |\Delta_k(x)|^A dx$ by majorizing the integral by discrete sums to which our large values estimate may be applied to bound the number of summands. Although our method will work for general $\Delta_k(x)$, the results are sharp only when $k = 2$ and $k = 3$, and therefore we shall consider only these cases. The basic estimate is the following

<u>THEOREM 10.8.</u> Let $1 \le t_1 < t_2 < \dots < t_R \le T$ and $|t_r - t_s| \ge V$ for $r \ne s \le R$. If $\Delta_2(t_r) \gg V > T^{7/32+\varepsilon}$ for $r \le R$, then

$$R \ll T^{\varepsilon}(TV^{-3} + T^{15/4}V^{-12}). \tag{10.54}$$

If $\Delta_3(t_r) \gg V > T^{18/67+\varepsilon}$ for $r \le R$, then

$$R \ll T^{\varepsilon}(T^2V^{-4} + T^{57/13}V^{-132/13}). \tag{10.55}$$

Another proof of $\alpha_2 \le 1/3$, $\alpha_3 \le 1/2$ follows at once from Theorem 10.8 if we take $R = 1$, $t_R = T = x$, though the above estimates are naturally of greater interest if R is assumed to be large in some sense. Also we could have formulated Theorem 10.8 with the spacing condition $|t_r - t_s| \ge 1$ for $r \ne s \le R$, and the only change would be that the exponent of V in (10.54) and (10.55) would be increased by unity. However the spacing condition $|t_r - t_s| \ge V$ ($r \ne s$) imposed in the theorem seems more appropriate, since by an argument analogous to (10.38) and (10.39) we have $|\Delta_k(t')| \ge V/2$ if $|\Delta_k(t)| \ge V$ and $|t' - t| \le Vt^{-\varepsilon}$.

Now we suppose that A is a fixed positive number (not necessarily an integer), and we formulate our power moment results for $\Delta_k(x)$ in the next two theorems.

THEOREM 10.9

$$\int_1^T |\Delta_2(t)|^A dt \ll T^{(A+4+\varepsilon)/4}, \qquad 0 \le A \le 35/4, \tag{10.56}$$

$$\int_1^T |\Delta_2(t)|^A dt \ll T^{(35A+38+\varepsilon)/108}, \qquad A \ge 35/4. \tag{10.57}$$

THEOREM 10.10.

$$\int_1^T |\Delta_3(t)|^A dt \ll T^{(106A+253+\varepsilon)/279}, \qquad 2 \le A \le 2237/607 = 3.685\ldots \tag{10.58}$$

$$\int_1^T |\Delta_3(t)|^A dt \ll T^{(43A+63+\varepsilon)/96}, \qquad A \ge 2237/607. \tag{10.59}$$

Theorem 10.9 shows that in a mean sense $|\Delta_2(t)|$ is of the conjectured order $t^{1/4+\varepsilon}$ for much higher powers than only the second, which was previously known only. The ranges for A in (10.56) and (10.58) both depend on the best known values (10.13) and (10.14) for α_2 and α_3 respectively, and any improvement of these bounds for α_2 and α_3 would result in a wider range for A. The limit that (10.54) can theoretically give is

$$\int_1^T |\Delta_2(t)|^{11} dt \ll T^{15/4+\varepsilon}, \tag{10.60}$$

and this would in turn imply the (yet hypothetical) estimate $\alpha_2 \le 5/16$ which improves on (10.13) and differs from the best possible value $\alpha_2 = 1/4$ by 1/16. To see how $\alpha_2 \le 5/16$ follows from (10.60) suppose that for some $\varepsilon > 0$ we have

$|\Delta(T)| = |\Delta_2(T)| > T^{5/16+\varepsilon} = G$. If $|t - T| \le GT^{-\delta}$ $(\delta > 0)$, then as in (10.39) we have $|\Delta(t)| \ge G/2$, and so by Hölder's inequality and (10.10) we obtain

$$G^2T^{-\delta} \le \int_{T-GT^{-\delta}}^{T+GT^{-\delta}} |\Delta(t)|\,dt \le$$

$$\left(\int_{T-GT^{-\delta}}^{T+GT^{-\delta}} |\Delta(t)|^{11}dt\right)^{1/11}(2GT^{-\delta})^{10/11} \ll T^{(15/4+\varepsilon)/11}G^{10/11}T^{-10\delta/11},$$

and this is a contradiction if δ is sufficiently small, in particular if $0 < \delta < 11\varepsilon$. This implies $\alpha_2 \le 5/16$, and the best unconditional estimate of α_2 which follows from (10.56) by this method of proof is only slightly weaker than (10.13).

Proof of Theorem 10.8. We start from (10.53) and use the Halász-Montgomery inequality to remove $d_k(n)$ from the sum in (10.53) and investigate the occurrence of large values of $\Delta_k(x)$. We use (1.36) and take $\xi = \{\xi_n\}_{n=1}^{\infty}$ with $\xi_n = d_k(n)n^{-(k+1)/2k}$ for $M < n \le 2M$ and zero otherwise, and we let $\varphi_r = \{\varphi_{r,n}\}_{n=1}^{\infty}$ with $\varphi_{r,n} = e(k(nt_r)^{1/k})$ for $M < n \le 2M$ and zero otherwise, where M is fixed and its range will be specified in a moment. We may restrict ourselves to the estimation of the number of points t_r lying in $[T/2, T]$ and we suppose that this interval is divided into subintervals of length not exceeding T_o $(\gg V)$. Denoting then by R_o the number of t_r's lying in an interval of length not exceeding T_o we have

$$R \ll R_o(1 + T/T_o), \tag{10.62}$$

since for $T \ge T_o$ we have $R \le R_o$. The idea behind this procedure (used already in Chapter 7 and Chapter 9) is that $|t_r - t_s| \le T_o$ for each R_o^2 pairs of points (t_r, t_s), so that a suitable choice of T_o will lead to (10.54) and (10.55). Choosing in (10.53) $N = T^{k-1+\varepsilon}V^{-k}$ we obtain by (1.36)

$$R_oV^2 \ll T^{(k-1)/k}\log T \max_{M\le N/2} \sum_{r\le R_o} \Big|\sum_{M<n\le 2M} d_k(n)n^{-(k+1)/2k}e(k(nt_r)^{1/k})\Big|^2 \ll \tag{10.63}$$

$$(TN)^{(k-1+\varepsilon)/k} + T^{(k-1+\varepsilon)/k} \max_{M\le N/2}\ \max_{r\le R_o} M^{-1/k} \sum_{s\le R_o, s\ne r} \Big|\sum_{M<n\le 2M} e(kn^{1/k}(t_r^{1/k} - t_s^{1/k}))\Big|,$$

since the contribution of the terms with $r = s$ is clearly $\ll N^{(k-1)/k}$. The last sum above is an exponential sum of the form

(10.64) $$S = \sum_{M<n\le 2M} e(f(n)),\ f(x) = kx^{1/k}(t_r^{1/k} - t_s^{1/k}),\ r \ne s,\ M \le x \le 2M,$$

which is very similar to the sum (7.18) in Lemma 7.1, only (10.64) is somewhat simpler. Since $f'(x)$ is monotonic for $M \le x \le 2M$ we may suppose that $|f'(x)| < 1$ or $|f'(x)| \ge 1$ by splitting S into two subsums if necessary. If $|f'(x)| < 1$ holds we estimate S by Lemma 2.5 and Lemma 2.1 as $S \ll \max_{M\le x\le 2M} |f'(x)|^{-1}$ to obtain

$$M^{-1/k} \max_{r\le R_0} \sum_{s\le R_0, s\ne r} |S| \ll M^{1-2/k} \max_{r\le R_0} \sum_{s\le R_0, s\ne r} |t_r^{1/k} - t_s^{1/k}|^{-1} \ll$$

(10.65)
$$N^{(k-2)/k}T^{(k-1)/k} \max_{r\le R_0} \sum_{s\le R_0, s\ne r} |t_r - t_s|^{-1} \ll N^{(k-2)/k}T^{(k-1)/k}V^{-1}\log T,$$

since $T/2 \le t_r \le T$ and $|t_r - t_s| \ge V$ for $r \ne s \le R_0$.

If $f'(x) \gg 1$ holds for $M \le x \le 2M$, then observing that

$$f^{(m)}(x) \asymp |t_r - t_s| T^{(1-k)/k} M^{(1-mk)/k},\ m = 1,2,\dots,$$

it follows that we may use the theory of exponent pairs (§3 of Chapter 2). Thus if $F = \max_{M\le x\le 2M} |f'(x)|$ and (p,q) is an exponent pair we have

$$S \ll F^pM^q \ll T_0^pT^{(p-pk)/k}M^{(qk+p-pk)/k}.$$

Therefore the contribution of these S is

$$\max_{M\le N/2} M^{-1/k} \max_{r\le R_0} \sum_{s\le R_0, s\ne r} |S| \ll$$

(10.66)
$$R_0 \max_{M\le N/2} M^{-1/k}M^{(qk+p-pk)/k}T_0^pT^{(p-pk)/k} \ll R_0T_0^pT^{(p-pk)/k}N^{(qk+p-1-pk)/p},$$

provided that

(10.67) $$qk \ge 1 + (k-1)p.$$

If this condition is satisfied, then from (10.63), (10.65) and (10.66) we obtain

(10.68) $$R_0V^2 \ll (TN)^{(k-1+\epsilon)/k} + T^{(2k-2+\epsilon)/k}N^{(k-2)/k}V^{-1} +$$

$$+ \quad R_o T_o^{p} T^{(k-1+\varepsilon)(1-p)/k} N^{(qk+p-1-pk)/k}.$$

Now we consider the case $k = 2$ where we choose $N = T^{k-1+\varepsilon}V^{-k} = T^{1+\varepsilon}V^{-2}$, and then the first two terms on the right-hand side of (10.68) are equal. We take $(p,q) = (4/18, 11/18)$ and note that with this exponent pair equality holds in (10.67) for $k = 2$, hence

$$(10.69) \qquad R_o \ll T^{1+\varepsilon}V^{-3} + R_o T_o^{2/9} T^{7/18+\varepsilon} V^{-2}.$$

Choosing $T_o = V^9 T^{-7/4-\varepsilon}$ we have $T_o \gg V$ for $V > T^{7/32+\varepsilon}$, and (10.69) gives

$$(10.70) \qquad R_o \ll T^{1+\varepsilon}V^{-3},$$

hence (10.54) follows from (10.62) with $T_o = V^9T^{-7/4-\varepsilon}$.

If $k = 3$ we choose $N = T^{k-1+\varepsilon}V^{-k} = T^{2+\varepsilon}V^{-3}$ and in (10.68) we take $(p,q) = (13/40, 22/40)$. With this exponent pair equality holds in (10.67) for $k = 3$, and (10.68) gives

$$(10.71) \qquad R_o \ll T^{2+\varepsilon}V^{-4} + R_o T_o^{13/40} T^{18/40+\varepsilon} V^{-2}.$$

Choosing $T_o = V^{80/13}T^{-18/13-\varepsilon}$ we have $T_o \gg V$ for $V > T^{18/67+\varepsilon}$, hence (10.71) gives $R_o \ll T^{2+\varepsilon}V^{-4}$ and (10.55) follows again from (10.62). With a little more care we could replace T^{ε} in (10.54) and (10.55) by a suitable log-power, and very small improvements in the second terms on the right-hand sides of (10.54) and (10.55) could be obtained by a more elaborate choice of the exponent pair (p,q). From (10.68) one obtains for general k the estimate $R \ll T^{k-1+\varepsilon}V^{-k-1}$ for $|\Delta_k(t_r)| \ge V = V(T,k)$, $r \le R$, but in view of $\alpha_4 \le 1/2$ (Theorem 10.2) this is weak already for $k = 4$.

Proof of Theorem 10.9 and Theorem 10.10. It is sufficient to prove our estimates for integrals over $[T/2,T]$ and then to sum over intervals of the form $[2^{-m}T, 2^{1-m}T]$, $m \ge 1$. We denote by τ_r the point for which

$$|\Delta_2(\tau_r)| = \max_{t \in [T/2+r-1,\, T/2+r]} |\Delta_2(t)|, \quad r = 1,2,\ldots$$

and we consider first those τ_r for which

$$T^{1/4} \le 2^m = V \le |\Delta_2(\tau_r)| < 2V.$$

There are $O(\log T)$ choices for V ($\ll T^{35/108+\varepsilon}$ by (10.13)), and by picking the maximal $|\Delta_2(\tau_r)|$ in τ-intervals of length V and by considering separately points with even and odd indexes we may construct a system of points which we shall label $t_1, t_2, \ldots, t_R$, $R = R(V)$ and which satisfy

$$(10.72)\qquad T^{1/4} \le 2^m = V \le |\Delta_2(t_r)| < 2V, \quad |t_r - t_s| \ge V \quad \text{for } r \ne s \le R = R(V),$$

so that we may write

$$(10.73)\qquad \int_{T/2}^{T} |\Delta_2(t)|^A dt \le T^{(A+4+\varepsilon)/4} + \sum_V V \sum_{r\le R(V)} |\Delta_2(t_r)|^A.$$

Now we consider the range $2 \le A \le 11$ and we use (10.54) to bound $R = R(V)$, keeping in mind that (10.72) holds. Using $V \ll T^{35/108+\varepsilon}$ we obtain

$$(10.74)\qquad V \sum_{r\le R(V)} |\Delta_2(t_r)|^A \ll RV^{A+1} \ll T^{\varepsilon}(TV^{A-2} + T^{15/4}V^{A-11}) \ll$$

$$\ll T^{1+35(A-2)/108+\varepsilon} + T^{(A+4+\varepsilon)/4}.$$

Here the first term is larger than the second for $35/4 \le A \le 11$, while the second is larger for $2 \le A \le 35/4$, which in view of (10.73) proves (10.56) for $2 \le A \le 35/4$, while the estimate for $0 \le A < 2$ follows easily by Hölder's inequality for integrals and the estimate for $A = 2$. To obtain (10.57) for $A > 11$ we proceed analogously, only now we have

$$V \sum_{r\le R(V)} |\Delta_2(t_r)|^A \ll T^{\varepsilon}(TV^{A-2} + T^{15/4}V^{A-11}) \ll$$

$$T^{1+35(A-2)/108+\varepsilon} + T^{15/4+35(A-11)/108+\varepsilon} + T^{(A+4+\varepsilon)/4} \ll T^{(35A+38+\varepsilon)/108}.$$

The proof of Theorem 10.10 is similar to the proof of Theorem 10.9 and utilizes (10.55) and (10.14), but while the proof of Theorem 10.9 is independent of (10.29), the proof of Theorem 10.10 will require a weak form of (10.46), namely $\int_1^T \Delta_3^2(t)dt \ll T^{5/3+\varepsilon}$. This last bound may be obtained directly from (10.53) with $k = 3$, $N = T$ by squaring the modulus and integrating termwise. Instead of (10.73) we impose for the proof of Theorem 10.10 a similar condition, namely

$$U \le 2^m = V \le |\Delta_3(t_r)| < 2V, \quad |t_r - t_s| \ge V \text{ for } r \ne s \le R = R(V),$$

where the points t_r are constructed analogously as in the previous case and the optimal choice for U is $U = T^{106/279}$. For $2 \le A \le 2237/607 = 3.6853\ldots$ we have then

$$(10.75)\qquad \int_{T/2}^{T} |\Delta_3(t)|^A dt \ll T^{5/3+\varepsilon}U^{A-2} + \sum_{V\ge U} V \sum_{r\le R(V)} |\Delta_3(t_r)|^A,$$

$$V\sum_{r\le R(V)} |\Delta_3(t_r)|^A \ll RV^{A+1} \ll T^{\varepsilon}(T^2V^{A-3} + T^2U^{A-3} + T^{57/13}U^{A-119/13}) \ll$$

$$T^{(43A+63+\varepsilon)/96} + T^{(106A+240+\varepsilon)/279} + T^{(106A+253+\varepsilon)/279}.$$

Here the third term is the largest one for $2 \le A \le 2237/607$ and (10.75) gives

$$\int_{T/2}^{T} |\Delta_3(t)|^A dt \ll T^{5/3+106(A-2)/279+\varepsilon} + T^{(106A+253+\varepsilon)/279} \ll$$

$$\ll T^{(106A+352+\varepsilon)/279}.$$

This proves (10.58). For $A > 2237/607$ the analysis is analogous and gives (10.59).

§8. The circle problem

This chapter is concluded by a discussion of the classical circle problem, which has been mentioned in §4 of Chapter 3. The problem of the estimation of

$$P(x) = R(x) - \pi x = \sum_{n\le x} r(n) - \pi x$$

bears many resemblances to the estimation of $\Delta(x)$ in the divisor problem, and we recall G.H. Hardy's classical formula ((3.36) with $q = 1$)

$$(10.76)\qquad {\sum_{n\le x}}' r(n) = \pi x - 1 + x^{1/2}\sum_{n=1}^{\infty} r(n)n^{-1/2}J_1(2\pi\sqrt{nx}),$$

or using the approximation

$$J_1(y) = -(2/\pi y)^{1/2}\cos(y + \pi/4) + O(y^{-3/2})$$

we may write

$$(10.77)\qquad P(x) = O(x^{\varepsilon}) - \pi^{-1}x^{1/4}\sum_{n=1}^{\infty} r(n)n^{-3/4}\cos(2\pi\sqrt{nx} + \pi/4),$$

since in (10.76) we have to count $r(x)/2$ if $n = x$ is an integer, and obviously $r(n) \ll n^{\varepsilon}$. A trivial bound for $P(x)$ is $P(x) \ll x^{1/2}$, since $P(x)$ is clearly majorized by the circumference of a circle with radius $x^{1/2}$. One would expect that (10.77) would provide the analogue of the truncated Voronoï formula (3.16) for $\Delta(x)$, and this would be

(10.78) $$P(x) = -\pi^{-1}x^{1/4}\sum_{n\le N} r(n)n^{-3/4}\cos(2\pi\sqrt{nx} + \pi/4) + O(x^{\varepsilon}(1 + (x/N)^{1/2})).$$

A direct proof of (10.78) via (10.76) does not seem easy (as is also the case in the analogous problem of the truncated formula for $\Delta(x)$), but one may use the method of Titchmarsh's proof of (3.16) by considering $\sum_{n=1}^{\infty} r(n)n^{-s}$ for $\mathrm{Re}\, s > 1$ and using the truncated Perron formula (1.10) to estimate $\sum_{n\le x} r(n)$. A similar approach has been adopted by H.-E. Richert [2], where general estimates for sums of the type $\sum_{n\le x} f(n)(x - n)^{\varkappa}$ are considered for certain classes of arithmetical functions f, without developing the sums in question into infinite series containing (generalized) Bessel functions, but into explicit exponential sums of length N plus error terms, and this is exactly what is needed for (10.78). Therefore instead of trying to obtain a direct proof of (10.78), we shall briefly state now Richert's discussion of the circle problem, and then obtain a result (Lemma 10.5) which is analogous to (10.78) and may be used to obtain estimates of power moments with P(x). Richert [2] transforms the circle problem into a divisor problem by noting that

(10.79) $$r(n) = 4\sum_{d|n,\, d\equiv 1 (\mathrm{mod}\, 2)} (-1)^{(d-1)/2},$$

and writing

(10.80) $$D(x;k_1,l_1,k_2,l_2) = \sum_{n_1n_2\le x,\, n_j\equiv l_j (\mathrm{mod}\, k_j)} 1 =$$

$$x(k_1k_2)^{-1}\log(x/k_1k_2) - \left(\frac{\Gamma'}{\Gamma}(l_1/k_1) + \frac{\Gamma'}{\Gamma}(l_2/k_2)\right)(k_1k_2)^{-1}x + \Delta(x;k_1,l_1,k_2,l_2)$$

one has

(10.81) $$R(x) = 4D(x;4,1,1,1) - 4D(x;4,3,1,1) = \pi x + 4\Delta(x;4,1,1,1) - 4\Delta(x;4,3,1,1),$$

when one recalls that $\frac{\Gamma'}{\Gamma}(3/4) - \frac{\Gamma'}{\Gamma}(1/4) = \pi$. Thus (10.81) shows that P(x) may be considered as the difference of two divisor problem error terms, and Richert [2] obtained

(10.82) $$\Delta(x;k_1,l_1,k_2,l_2) = (\sqrt{2}\pi)^{-1}(x/k_1k_2)^{1/4}\mathrm{Re}\left\{e(-1/8)\sum_{1\le n_1n_2\le N}(n_1n_2)^{-3/4}e(F)\right\} +$$

$$+ O((xN)^{1/5+\varepsilon}) + O((x/N)^{1/2+\varepsilon}),$$

where

$$(10.83) \qquad F = (4xn_1n_2/k_1k_2)^{1/2} - (l_1n_1)/k_1 - (l_2n_2)/k_2.$$

Here for $N \le x^{3/7}$ we have $x^{1/5}N^{1/5} \le x^{1/2}N^{-1/2}$, so that for the range $x^{1/3} \le N \le x^{3/7}$ the first error term in (10.82) may be discarded. Except for the linear part $-(l_1n_1)/k_1 - (l_2n_2)/k_2$ the exponential term in (10.82) is (up to a constant) the same as in the formula (3.17) for $\Delta(x)$. It is readily seen that the linear part poses no problem in the application of Kolesnik's method (Lemma 6.3), as the linear terms are small when compared with $(xn_1n_2)^{1/2}$, and moreover the linear terms vanish already in the second partial derivatives of $F = F(n_1,n_2)$ in (10.82). As in the proof of Theorem 10.1 we have then

<u>THEOREM 10.11.</u>

$$(10.84) \qquad R(x) = \sum_{n\le x} r(n) = \pi x + O(x^{35/108+\epsilon}).$$

Next to obtain (10.78) note that using (10.82) in (10.81) we have

$$e(-(l_1n_1)/k_1-(l_2n_2)/k_2) = \exp(-2\pi i l_1n_1/k_1) = \begin{cases} e^{-\pi i n_1/2}, & k_1 = 4, k_2 = l_1 = 1 \\ e^{\pi i n_1/2}, & k_1 = 4, k_2 = 1, l_1 = 3. \end{cases}$$

This shows that for $x^{1/3} \le N \le x^{3/7}$ we have

$$(10.85) \qquad P(x) = 4(\sqrt{2}\pi)^{-1}(x/4)^{1/4}\mathrm{Re}\Big\{e(-1/8)\sum_{n_1n_2\le N}(n_1n_2)^{-3/4}(-2i\sin(n_1\pi/2))e(\sqrt{xn_1n_2})\Big\}+$$

$$+ O((x/N)^{1/2+\epsilon}) = -4\pi^{-1}x^{1/4}\mathrm{Re}\Big\{e(1/8)\sum_{\substack{n_1n_2\le N\\ n_1\equiv 1 (\mathrm{mod} 2)}}(n_1n_2)^{-3/4}(-1)^{(n_1-1)/2}e(\sqrt{xn_1n_2})\Big\} +$$

$$+ O((x/N)^{1/2+\epsilon}) = -\pi^{-1}x^{1/4}\sum_{n\le N} r(n)n^{-3/4}\cos(2\pi\sqrt{xn} + \pi/4) + O((x/N)^{1/2+\epsilon}),$$

where we used (10.79). In view of the error term in (10.82) the best that this approach can give is $P(x) \ll x^{2/7+\epsilon}$, which though better than Theorem 10.11 is still poorer than the conjectured estimate $P(x) \ll x^{1/4+\epsilon}$. We would like to use (10.78) to obtain a result analogous to Theorem 10.9 for power moments of $P(x)$, but as we have (10.78) for $x^{1/3} \le N \le x^{3/7}$ we would not obtain a result of the same strength as Theorem 10.9, since we need (10.78) in the range $x^{1/3} \le N \le x^{1/2}$ for that purpose. Therefore we turn back again to Hardy's formula (10.76) and use the technique of exponential averaging, as introduced in Chapter 6, to obtain a result similar to (10.78), but without the restriction $N \le x^{3/7}$. This is

<u>Lemma 10.5</u>. For $T \le x \le 2T$, $T^{1/4} \le G \le T^{1/3}$ we have uniformly in x

$$P(x) \ll T^{\varepsilon}G + T^{1/4}\Big|\sum_{n\le TG^{-2}\log^2 T} r(n)n^{-3/4}\exp(2\pi i\sqrt{nx} - \tfrac{1}{4}\pi^2 nG^2x^{-1})\Big|. \tag{10.86}$$

<u>Proof of Lemma 10.5</u>. Let $\|x\|$ denote the distance of x to the nearest integer and let the hypotheses of the lemma hold. The first step in the proof will be to show that

$$x^{1/4}\sum_{n>T^2} r(n)n^{-3/4}e(\sqrt{nx}) \ll T^{\varepsilon}G, \quad \text{if } \|x\| \gg GT^{-3/4}. \tag{10.87}$$

To see this write

$$x^{1/4}\sum_{n>T^2} r(n)n^{-3/4}e(\sqrt{nx}) = x^{1/4}\int_{T^2}^{\infty} t^{-3/4}e(\sqrt{xt})dR(t) =$$

$$\pi x^{1/4}\int_{T^2}^{\infty} t^{-3/4}e(\sqrt{xt})dt + x^{1/4}\int_{T^2}^{\infty} t^{-3/4}e(\sqrt{xt})dP(t) =$$

$$o(1) + x^{1/4}\int_{T^2}^{\infty} t^{-3/4}e(\sqrt{xt})dP(t).$$

Integrating by parts and using $P(t) \ll t^{1/2}$ it is seen that the last expression above is

$$O(x^{-1/4}) - x^{1/4}\int_{T^2}^{\infty} P(t)(-\tfrac{3}{4}t^{-7/4} + \pi i x^{1/2}t^{-5/4})e(\sqrt{xt})dt = O(x^{-1/4}) - \pi i x^{3/4}I,$$

where

$$I = \int_{T^2}^{\infty} P(t)t^{-5/4}e(\sqrt{xt})dt. \tag{10.88}$$

Using (10.77) we obtain

$$I = O(T^{\varepsilon-1/2}) - \pi^{-1}\sum_{n=1}^{\infty} r(n)n^{-3/4}\int_{T^2}^{\infty} t^{-1}\cos(2\pi\sqrt{nt} + \pi/4)e(\sqrt{xt})dt.$$

The above integral is written as a sum of integrals of the type

$$\int_M^{2M} g(x)e^{iF(x)}dx \ll \max_{x\in[M,2M]}|g(x)| \max_{x\in[M,2M]}|F'(x)|^{-1}$$

by Lemma 2.1, eq. (2.3) to give

$$I \ll T^{\varepsilon-1/2} + T^{-1}\sum_{n=1}^{\infty} r(n)n^{-3/4}|n^{1/2} - x^{1/2}|^{-1} \ll$$

$$T^{\varepsilon-1/2} + T^{-1}\Big(\sum_{n\le x/2} + \sum_{x/2<n\le 2x} + \sum_{n>2x}\Big) \ll$$

$$T^{\varepsilon-1/2} + T^{\varepsilon-1}\sum_{x/2<n\le 2x} n^{-3/4}T^{1/2}|n - x|^{-1} \ll$$

$$T^{\varepsilon-1/2} + T^{\varepsilon-1/2}G^{-1} \ll T^{\varepsilon-1/2},$$

and this proves (10.87), since $G \ge T^{1/4}$. This step in the proof was necessary, since it reduces the series in (10.77) to a finite expression, and thus serves as a basis for (10.86).

The next step is to derive a suitable averaged expression for $P(x)$, using the elementary integral $\int_{-\infty}^{\infty} \exp(-x^2)dx = \pi^{1/2}$. Abbreviating $L = \log T$ we have

(10.89) $$\pi^{1/2}P(x) - G^{-1}\int_{x-GL}^{x+GL} P(t)e^{-(x-t)^2G^{-2}}dt = o(1) + G^{-1}\int_{x-GL}^{x+GL}(P(x)-P(t))e^{-(x-t)^2G^{-2}}dt$$

$$\ll 1 + L\max_{|x-t|\le GL}|P(x) - P(t)| \ll GT^{\varepsilon},$$

since

$$|P(x) - P(t)| \le \pi|x - t| + \Big|\sum_{t\le n\le x} r(n)\Big| + O(T^{\varepsilon}) \ll (1 + |x - t|)T^{\varepsilon},$$

in view of $r(n) \ll n^{\varepsilon}$. To use (10.87) write

$$\int_{x-GL}^{x+GL} P(t)e^{-(x-t)^2G^{-2}}dt = \int_{x-GL}^{x+GL}\Big(P(t) + \pi^{-1}t^{1/4}\sum_{n\le T^2} r(n)n^{-3/4}\cos(2\pi\sqrt{nt}+\pi/4)\Big)e^{-(x-t)^2G^{-2}}dt$$

(10.90)
$$-\pi^{-1}\int_{x-GL}^{x+GL} t^{1/4}\sum_{n\le T^2} r(n)n^{-3/4}\cos(2\pi\sqrt{nt}+\pi/4)e^{-(x-t)^2G^{-2}}dt = I_1 - \pi^{-1}I_2,$$

say. The main contribution in (10.90) (and hence the main contribution to $P(x)$ in (10.89)) comes from I_2, and to show this we shall prove

(10.91) $$I_1 \ll G^2T^{\varepsilon}.$$

To achieve this let

(10.92) $$A_1 = \bigcup_{n=1}^{\infty}\Big\{[x - GL, x + GL] \cap [n - GT^{-3/4}, n + GT^{-3/4}]\Big\},\ A_2 = [x-GL, x+GL] \setminus A_1,$$

and split I_1 into integrals over A_1 and A_2 respectively. For $t \in A_1$ we shall use the trivial

$$P(t) + \pi^{-1}t^{1/4}\sum_{n\le T^2} r(n)n^{-3/4}\cos(2\pi\sqrt{nt} + \pi/4) \ll T^{3/4+\varepsilon}$$

to obtain

$$\int_{A_1} \ll GT^{3/4+\varepsilon}GT^{-3/4} \ll G^2T^{\varepsilon},$$

since obviously $|A_1| \ll G^2T^{-3/4+\varepsilon}$. For the integral over A_2 we use (10.77) and (10.87) to obtain at once $\int_{A_2} \ll G^2T^{\varepsilon}$, hence (10.91).

Therefore we are left with the evaluation of

$$(10.93) \qquad I_2 = \int_{-GL}^{GL} (x+t)^{1/4}\sum_{n\le T^2} r(n)n^{-3/4}\cos(2\pi\sqrt{n(x+t)} + \pi/4)\exp(-t^2G^{-2})\,dt,$$

and we can replace $(x+t)^{1/4}$ by $x^{1/4}$ with an error which is $\ll G^2T^{\varepsilon}$, so that combining previous estimates we have

$$(10.94) \quad P(x) \ll GT^{\varepsilon} + G^{-1}T^{1/4}\Big|\sum_{n\le T^2} r(n)n^{-3/4}\int_{-GL}^{GL}\exp(2\pi i\sqrt{n(x+t)} - t^2G^{-2})\,dt\Big|.$$

Using (1.34) and Taylor's formula we obtain

$$\int_{-GL}^{GL}\exp(2\pi i\sqrt{n(x+t)} - t^2G^{-2})\,dt = e(\sqrt{nx})\int_{-\infty}^{\infty}\exp(\pi i t x^{-1/2}n^{1/2} - \tfrac{1}{4}\pi i t^2x^{-3/2}n^{1/2} - t^2G^{-2})\,dt$$

$$+ O(G^4L^4T^{-5/2}n^{1/2}) = (\pi/Y)^{1/2}\exp(2\pi i\sqrt{nx} - \tfrac{1}{4}\pi^2nY^{-1}x^{-1}) + O(G^4L^4T^{-5/2}n^{1/2}),$$

where we have set

$$Y = G^{-2} + \tfrac{1}{4}\pi i x^{-3/2}n^{1/2}.$$

Therefore from (10.94) it follows that

$$(10.95) \quad P(x) \ll GT^{\varepsilon} + G^{-1}T^{1/4}|Y|^{-1/2}\Big|\sum_{n\le T^2} r(n)n^{-3/4}\exp(2\pi i\sqrt{nx} - \tfrac{1}{4}\pi^2nY^{-1}x^{-1})\Big| + G^3T^{\varepsilon-3/4}.$$

Now for $n \ge TG^{-2}L^2$ and any fixed $c > 0$

$$\left|\exp(-\tfrac{1}{4}\pi^2nY^{-1}x^{-1})\right| \le \exp(-\tfrac{1}{4}\pi^2nx^{-1}G^2) \le \exp(-\pi^2L^2/4) \ll T^{-c},$$

so that the contribution of the terms with $n > TG^{-2}L^2$ in (10.95) is negligible and moreover the last term in (10.95) may be discarded since $G^3T^{-3/4} < G$. For $n \le TG^{-2}L^2$ one can replace Y by G^{-2} with a total error $\ll GT^{\varepsilon}$, hence (10.86) follows from (10.95).

Lemma 10.5 is now completely analogous to the truncated Voronoi formula (3.17) (with $G = (TN^{-1}\log^2 T)^{1/2}$), since the exponential factor $\exp(-\pi^2 nG^2x^{-1}/4)$ which appears in (10.86) is ≤ 1 and does not affect order results obtainable from (10.86). Combining Theorem 10.11 with Lemma 10.5 one can obtain easily the analogue of Theorem 10.9 virtually by repeating the same proof with d(n) replaced by r(n), and the result will be

THEOREM 10.12.

$$\int_1^T |P(x)|^A dx \ll T^{(A+4+\varepsilon)/4}, \qquad 0 \le A \le 35/4,$$

$$\int_1^T |P(x)|^A dx \ll T^{(35A+38+\varepsilon)/108}, \qquad A > 35/4.$$

This is new for $A > 2$, while a sharp asymptotic formula for $A = 2$, completely analogous to (10.35), has been established by K.-C. Tong [2]. The estimates (10.85) and (10.86) show great similarities in the circle and divisor problem, and naturally the analogue of (10.86) may be derived by the method of Lemma 10.5 from the Voronoї series expression (3.1) for $\Delta(x)$.

N O T E S

Strictly speaking (10.3) holds for x not an integer, since in $\sum'_{n\le x} d_k(n)$ the last term is to be counted as $\frac{1}{2}d_k(x)$ if x is an integer, but as already remarked in Notes of Chapter 3 concerning $\Delta(x) = \Delta_2(x)$, for most purposes this distinction is irrelevant.

An explicit formula for γ_k, the k-th coefficient in the Laurent expansion of $\zeta(s)$ at $s = 1$, is given by (10.5). A simple proof of this formula (obtained already by Stieltjes in the 19th century) will be presented now. Let $k, r \ge 0$ be integers and let

$$c_r = -\int_{1-0}^{\infty} t^{-1}\log^r t \cdot d\psi(t), \quad \psi(t) = t - [t] - 1/2.$$

By the Stieltjes integral representation we have

$$c_r = -\lim_{N\to\infty} \int_{1-0}^{N} t^{-1}\log^r t \cdot d\psi(t) = \lim_{N\to\infty} \left(\int_{1-0}^{N} t^{-1}\log^r t \cdot d[t] - \int_{1-0}^{N} t^{-1}\log^r t \cdot dt \right)$$

$$= \lim_{N\to\infty} \left(\sum_{n\le N} n^{-1}\log^r n - (r+1)^{-1}\log^{r+1} N \right).$$

Let further $S_r(x) = \sum_{n\le x} n^{-1}\log^r x/n$. Then

$$S_r(x) = \int_{1-0}^{x} t^{-1}\log^r x/t . d[t] = \int_1^x t^{-1}\log^r x/t\cdot dt - \int_{1-0}^{x} t^{-1}\log^r x/t\cdot d\psi(t) =$$

$$(r+1)^{-1}\log^{r+1}x - \int_{1-0}^{\infty} t^{-1}(\log x - \log t)^r d\psi(t) + \int_x^{\infty} t^{-1}\log^r x/t\cdot d\psi(t) =$$

$$(r+1)^{-1}\log^{r+1}x + \sum_{k=0}^{r}(-1)^k\binom{r}{k}c_k\log^{r-k}x + R_r(x),$$

say. Our aim is to prove $\gamma_k = (-1)^k c_k/k!$ for $0 \le k \le r$, where for $\operatorname{Re} s > 0$

$$\zeta(s+1) = \sum_{n=1}^{\infty} n^{-s-1} = s^{-1} + \sum_{k=0}^{\infty}\gamma_k s^k.$$

To accomplish this, observe first that

$$\int_1^{\infty} t^{-s}dS_r(t) = r\int_1^{\infty} t^{-s-1}S_{r-1}(t)dt = -rs^{-1}\int_1^{\infty} S_{r-1}(t)dt^{-s} =$$

$$rs^{-1}\int_1^{\infty} t^{-s}dS_{r-1}(t) = \ldots = r!s^{-r}\int_1^{\infty} t^{-s}dS_o(t) = r!s^{-r}\sum_{n=1}^{\infty} n^{-s-1} =$$

$$r!s^{-r-1} + \sum_{k=0}^{\infty}\gamma_k r!s^{k-r}.$$

On the other hand, using the expression for $S_r(x)$, it is seen that

$$\int_1^{\infty} t^{-s}dS_r(t) = r!s^{-r-1} + \sum_{k=0}^{\infty} a_k s^{k-r},$$

where

$$a_k s^{k-r} = \int_1^{\infty}(-1)^k\binom{r}{k}c_k t^{-s}d(\log^{r-k}t) = (-1)^k\binom{r}{k}c_k(r-k)\int_1^{\infty} t^{-s-1}\log^{r-k-1}t\cdot dt =$$

$$s^{k-r}(-1)^k\binom{r}{k}c_k(r-k)\int_0^{\infty} e^{-v}v^{r-k-1}dv = s^{k-r}(-1)^k r!c_k/k!.$$

Comparing the two series expansions for $\int_1^{\infty} t^{-s}dS_r(t)$ we obtain the desired identity $\gamma_k = (-1)^k c_k/k!$ for $0 \le k \le r$, and since r may be arbitrary (10.5) follows.

The paper of A.F. Lavrik, M.I. Israilov and Ž. Edgorov [1] which contains the proof of (10.7) and (10.8) also gives an explicit evaluation of $\int_1^{\infty}\Delta_k(u)u^{-2}du$ for $k \le 5$ in terms of the γ_k's as defined by (10.5); for instance

$$\int_1^{\infty}\Delta_2(u)u^{-2}du = (\gamma - 1)^2 + 2\gamma_1.$$

Explicit evaluation of coefficients of $P_{k-1}(t)$ in (10.3) is also discussed by A.F. Lavrik [1].

The history of the estimation of α_k (and in particular of α_2) is at least as long and as rich as the history of estimates for $\zeta(1/2 + it)$, which was given in the Notes of Chapter 6. This history goes deep into the 19th century to P.G.L. Dirichlet, who proved in an elementary way that $\alpha_2 \le 1/2$ and in whose honour the problem is known as "the Dirichlet divisor problem". Further estimates for α_2 are as follows:

$\alpha_2 \le 1/3 = 0.333333\ldots,$	G.F. Voronoi [1], 1904
$\alpha_2 \le 33/100 = 0.33$	J.G. van der Corput [2], 1922
$\alpha_2 \le 27/82 = 0.329268\ldots,$	J.G, van der Corput [3], 1928
$\alpha_2 \le 15/46 = 0.326086\ldots,$	H.-E. Richert [1], 1953 and Chih Tsung-tao [1], 1950
$\alpha_2 \le 12/37 = 0.324324\ldots,$	G. Kolesnik [2], 1969
$\alpha_2 \le 346/1067 = 0.324273\ldots,$	G. Kolesnik [3], 1973
$\alpha_2 \le 35/108 = 0.324074\ldots,$	G. Kolesnik [6], 1982.

Kolesnik [6] obtains actually $\Delta_2(x) \ll x^{35/108+\varepsilon}$ (so that (10.13) is slightly sharper), but his argument clearly gives also (10.13). Anyway the log-factors are not so important as Kolesnik's method is not exhausted by the value $\alpha_2 \le 35/108$, and he has kindly informed me that the best it can give at present is a value slightly less than 35/108.

The history of estimates for α_3 is as follows:

$\alpha_3 \le 1/2 = 0.5$	G.H.Hardy and J.E.Littlewood [2], 1922
$\alpha_3 \le 43/87 = 0.494252\ldots,$	A. Walfisz [2], 1925
$\alpha_3 \le 37/75 = 0.493333\ldots,$	F.V. Atkinson [1], 1941
$\alpha_3 \le 14/29 = 0.482758\ldots,$	Yüh Ming-i [1], 1958
$\alpha_3 \le 8/17 = 0.470588\ldots,$	Yüh Ming-i and Wu Fang [1], 1962
$\alpha_3 \le 5/11 = 0.454545\ldots,$	Chen Jing-run [2], 1965
$\alpha_3 \le 43/96 = 0.447916\ldots,$	G. Kolesnik [5], 1981.

For several general estimates of α_k, all of which are poorer than those given by Theorem 10.2 when $k \ge 5$, the reader is referred to Chapter 12 of E.C. Titchmarsh [8].

The estimate $\alpha_k \le (3k-4)/4k$, $4 \le k \le 8$ of Theorem 10.2 has been given by D.R. Heath-Brown [8], who proved in [8] also $\alpha_k \le (k-3)/k$ for $k > 8$, but this is superseded now by corresponding estimates of Theorem 10.2, which is due to the author [2].

The method of proof of Theorem 10.2 is based on the use of Theorem 7.3 and shows that one can obtain $\alpha_k \le 1 - A/k$ for any fixed $A > 0$ and $k \ge k(A)$, but this is superseded by Theorem 10.3 for sufficiently large k. Besides choosing more carefully the exponent pairs in the proof of (7.65), there are other possibilities of improving Theorem 10.2 for large k, namely the bound $\alpha_k \le 1 - 34/(7k)$ for $k \ge 58$. Instead of $c(\theta) = (1 - \theta)/5$ $(5/6 \le \theta \le 1)$ one may use sharper bounds for $c(\theta)$ in (7.57) and Lemma 7.2 for appropriate ranges of θ. Thus from (6.59) with $l = 6$ we obtain $c(\theta) \le (1 - \theta)/6$ for $28/31 \le \theta \le 1$ and this will lead to $m(\sigma) \ge 5/(1 - \sigma)$ for $0.91 \le \sigma \le 1 - \varepsilon$, and consequently to $\alpha_k \le 1 - 5/k$ for $k \ge 58$. Still a better result may be obtained if in bounding $c(\theta)$ convexity is used for two consecutive values of l in (6.59). This was the idea used by A. Fujii [1], who obtained a bound for α_k which does not depend explicitly on k, but on a parameter b, so that additional calculations are necessary to evaluate α_k, and a general formula for α_k is difficult to obtain. A calculation shows that Fujii's estimates lead to better values than $\alpha_k \le 1 - 34/(7k)$ of Theorem 10.2 for $k \ge 109$, but his results are further improvable if instead of Theorem 7.10 of Titchmarsh [8] one uses sharper bounds for $m(\sigma)$ (when σ is close to 1) obtainable by the method of Theorem 7.3, as described above, Also a slight sharpening is possible if instead of (6.59) one uses the sharper bound

$$c(\theta) \le \frac{1}{4Q - 2}\cdot\frac{240Qq - 16Q + 128}{240Qq - 15Q + 128},\quad Q = 2^{q-1},\quad q = 3,4,\ldots,\quad \theta = 1 - \frac{q + 1}{4Q - 2},$$

which was proved long ago by E. Phillips [1].

Theorem 10.3 is due to H.-E. Richert [3], and was rediscovered by A.A. Karacuba [1], who in [2] proved a stronger result than Theorem 10.3, namely

$$\Delta_k(x) \ll x^{1-Ck^{-2/3}}(D\log x)^k,$$

where $C,D > 0$ are absolute constants and the estimate is uniform in k.

A. Fujii [1] showed that in Theorem 10.3 one may take

$$C = 2^{-1/2}(2^{3/2} - 1)^{-1/3}(39)^{-2/3},$$

improving the value of C given by A.A. Karacuba [2], but the value k_0 such that $\alpha_k \le 1 - Ck^{-2/3}$ for $k \ge k_0$ is not explicitly determined in either of these works.

Lemma 10.1 is a weakened form of a result of H.-E. Richert [4], who used I.M. Vinogradov's estimates [1], [2] (see also A. Walfisz [3] for a good account of Vinogradov's method) of exponential sums and gave an elegant proof of

$$\zeta(\sigma + it) \ll t^{100(1-\sigma)^{3/2}}(\log t)^{2/3}, \quad 1/2 \le \sigma \le 1, \ t \ge 2.$$

This result is significant when σ is close to 1, when it improves results obtainable by van der Corput's method.

Concerning Theorem 10.4 it may be mentioned that $\beta_2 = 1/4$, $\beta_3 = 1/3$ are classical results that may be found in Titchmarsh's book [8], while $\beta_4 = 3/8$ has been proved by Heath-Brown [8] and the remaining bounds of Theorem 10.4 are due to the author [3]. They improve on $\beta_5 \le 1/2$, $\beta_6 \le 35/62$, $\beta_7 \le 11/18$, $\beta_8 \le 149/230$ of K.-C. Tong [1]; indeed his bound for β_8 is poorer than our bound for β_7.

The form of Theorem 10.5 is due to H. Cramér [1], and curiously enough no result of this type is to be found in Titchmarsh [8]. The results of Theorem 10.6 and Theorem 10.7 are new and have not appeared in print yet, while the theorems of §7 are proved by the author in [5].

Theorem 10.5 and its analogue for the circle problem provide weak omega results for $\Delta(x)$ and $P(x)$, namely $\Delta(x) = \Omega(x^{1/4})$ and $P(x) = \Omega(x^{1/4})$. Some better results are known, and in 1916 G.H. Hardy [1], [2] proved

$$\Delta(x) = \begin{cases} \Omega_+((x\log x)^{1/4}\log\log x) \\ \Omega_-(x^{1/4}) \end{cases},$$

$$P(x) = \begin{cases} \Omega_-((x\log x)^{1/4}) \\ \Omega_+(x^{1/4}). \end{cases}$$

Hardy's Ω_- estimate for $\Delta(x)$ and Ω_+ estimate for $P(x)$ have been improved a little by K.S. Gangadharan [1], and the best results in this direction seem to be due to K. Corrádi and I. Kátai [1], who proved with some absolute

$C_1, C_2 > 0$ that

$$\Delta(x) = \Omega_-(x^{1/4}\exp(C_1(\log\log x)^{1/4}(\log\log\log x)^{-3/4})),$$

$$P(x) = \Omega_+(x^{1/4}\exp(C_2(\log\log x)^{1/4}(\log\log\log x)^{-3/4})).$$

Hardy's Ω_+ estimate for $\Delta(x)$ and Ω_- estimate for $P(x)$ withstood improvement for a very long time. Only recently J.L. Hafner [1] succeeded in proving with some absolute $C_3, C_4 > 0$ that

$$\Delta(x) = \Omega_+((x\log x)^{1/4}(\log\log x)^{(3+2\log 2)/4}\exp(-C_3(\log\log\log x)^{1/2})),$$

$$P(x) = \Omega_-((x\log x)^{1/4}(\log\log x)^{(\log 2)/4}\exp(-C_4(\log\log\log x)^{1/2})).$$

Although the circle problem, discussed in §8, is certainly a digression from the main topic which is the zeta-function, I have nevertheless felt it appropriate to include this material (new and hitherto unpublished) for two reasons. Firstly the results seem to be interesting, and secondly they stress the intrinsic connection between the divisor problem for $\Delta(x)$ and the circle problem. Most earlier authors have investigated the circle problem, divisor problem and the problem of the order of $\zeta(1/2 + it)$ separately and by different methods. The approach presented here shows a unified view of the circle and divisor problem, and the idea to use exponential averaging in the proof of Lemma 10.5 has been kindly suggested by M. Jutila, whose works [4] and [5] (parts of which will be discussed in the next chapter) show the intrinsic connection between $\Delta(x)$, $\zeta(1/2 + it)$ and $E(T)$ mainly in the light of Atkinson's formula. The problem of the estimation of $\zeta(1/2 + it)$ was already discussed in Chapter 6 in Theorem 6.3, which bears a close resemblance to Theorem 10.1. It turns out at present that all the best known exponents in the divisor problem, circle problem and the problem of the order of $E(T)$ are the same one, namely $35/108 + \varepsilon$ by Kolesnik's method. Whether the real order of the functions in question (for which one naturally conjectures the exponent $1/4 + \varepsilon$) is the same (up to ε's and log-factors) is not possible to tell yet, though one expects the answer to be affirmative.

Aleksandar Ivić

TOPICS IN RECENT ZETA FUNCTION-THEORY

CHAPTER 11

ATKINSON'S FORMULA FOR THE MEAN SQUARE

CHAPTER 11

ATKINSON'S FORMULA FOR THE MEAN SQUARE

§1. Introduction

A classical problem in zeta-function theory is the investigation of the asymptotic behaviour of the integral

$$I(T) = \int_0^T |\zeta(1/2 + it)|^2 dt,$$

and the first non-trivial result has been obtained by G.H. Hardy and J.-E. Littlewood [1], where they showed that

$$I(T) = (1 + o(1))T\log T. \tag{11.1}$$

A substantial advance in this problem has been made in 1922 by J.E. Littlewood [1] who proved that

$$E(T) \ll T^{3/4+\varepsilon}, \tag{11.2}$$

where

$$E(T) = I(T) - T\log(T/2\pi) - T(2\gamma - 1). \tag{11.3}$$

An explicit formula for $E(T)$ was discovered by F.V. Atkinson [3], and this formula is the main topic of this chapter. This deep and important result of Atkinson seems to have been neglected for a long time, until first important applications have been made by D. R. Heath-Brown [1], [2], and it seems certain that the possibilities of Atkinson's formula are far from being exhausted. The depth and the scope of Atkinson's formula seem to provide an adequate ending of this text, and the result will be formulated as

THEOREM 11.1. Let $0 < A < A'$ be any fixed constants such that $AT < N < A'T$ and let $N' = N'(T) = T/2\pi + N/2 - (N^2/4 + NT/2\pi)^{1/2}$. Then

$$E(T) = 2^{-1/2}\sum_{n\le N}(-1)^n d(n)n^{-1/2}(\operatorname{arsinh}((\pi n/2T)^{1/2}))^{-1}(T/2\pi n+1/4)^{-1/4}\cos(f(T,n))$$
$$- 2\sum_{n\le N'} d(n)n^{-1/2}(\log T/2\pi n)^{-1}\cos(T(\log T/2\pi n) - T + \pi/4) + O(\log^2 T), \tag{11.4}$$

where

$$f(T,n) = 2T\operatorname{arsinh}((\pi n/2T)^{1/2}) + (2\pi nT + \pi^2 n^2)^{1/2} - \pi/4. \tag{11.5}$$

We may rewrite (11.4) in the form

$$(11.6)\qquad E(T) = \sum\nolimits_1(T) + \sum\nolimits_2(T) + O(\log^2 T),$$

$$(11.7)\qquad \sum\nolimits_1(T) = 2^{1/2}(T/2\pi)^{1/4}\sum_{n\le N}(-1)^n d(n)n^{-3/4}e(T,n)\cos(f(T,n)),$$

where

$$(11.8)\qquad e(T,n) = (1+\pi n/2T)^{-1/4}\left\{(2T/\pi n)^{1/2}\operatorname{arsinh}((\pi n/2T)^{1/2})\right\}^{-1} = 1 + O(nT^{-1}),$$

$$(11.9)\qquad \sum\nolimits_2(T) = -2\sum_{n\le N'} d(n)n^{-1/2}(\log T/2\pi n)^{-1}\cos(g(T,n)),$$

where

$$(11.10)\qquad g(T,n) = T\log(T/2\pi n) - T + \pi/4.$$

Using the Taylor expansion

$$(11.11)\qquad f(T,n) = -\pi/4 + 4\pi(nT/2\pi)^{1/2} + O(n^{3/2}T^{-1/2}),\ n = o(T),$$

it is seen that, apart from the oscillating factor $(-1)^n$, the first $o(T^{1/3})$ terms in $\sum_1(T)$ are asymptotically equal to the corresponding terms in the truncated Voronoï formula for $2\pi\Delta(T/2\pi)$, as given by (3.17). This deep analogy between the divisor problem and the mean square of the zeta-function on the critical line has been one of primary motivations of Atkinson's work concerning Theorem 11.1. This tópic will be further pursued in §5.

There is another possibility of proving an explicit formula for $E(T)$. This has been found recently by R. Balasubramanian [1] who used a complicated integration technique based on the classical Riemann-Siegel formula for the zeta-function to prove

$$(11.12)\qquad E(T) = 2\sum_{n\le K}\sum_{m\ne n\le K}\frac{\sin(T\log n/m)}{(mn)^{1/2}\log n/m} + 2\sum_{n\le K}\sum_{m\ne n\le K}\frac{\sin(2\theta - T\log mn)}{(mn)^{1/2}(2\theta' - \log mn)} + O(\log^2 T),$$

where

$$(11.13)\qquad \theta = \theta(T) = \frac{T}{2}\log(T/2\pi) - \frac{T}{2} - \pi/8,\ K = \left[(T/2\pi)^{1/2}\right].$$

Upper bounds for $E(T)$ may be obtained from (11.12), but it seems simpler to use Atkinson's formula and the averaging techniques similar to those of Chapter 6. In this way it will be seen that

(11.14) $$E(T) \ll T^{35/108+\varepsilon},$$

which is completely analogous to corresponding estimates for $\Delta(x)$ and $P(x)$ furnished by Theorem 10.1 and Theorem 10.11 respectively, since the estimation will be reduced to very similar exponential sums. We reserve §2 of this chapter for the proof of the difficult Theorem 11.1, while some applications of Atkinson's formula will be presented in later sections.

§2. Proof of Atkinson's formula

We start from the obvious identity, valid for $\mathrm{Re}\, u > 1$, $\mathrm{Re}\, v > 1$,

(11.15) $$\zeta(u)\zeta(v) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-u}n^{-v} = \zeta(u+v) + f(u,v) + f(v,u),$$

where

(11.16) $$f(u,v) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r^{-u}(r+s)^{-v}.$$

We shall show first that $f(u,v)$ is a meromorphic function of u and v for $\mathrm{Re}(u+v) > 0$. Taking $\mathrm{Re}\, v > 1$ and writing $\psi(x) = x - [x] - 1/2$, $\psi_1(x) = \int_1^x \psi(t)dt$, it follows on integrating by parts that

$$\sum_{s=1}^{\infty}(r+s)^{-v} = \int_{1-0}^{\infty}(r+t)^{-v}d[t] = v\int_r^{\infty}([x]-r)x^{-v-1}dx =$$

$$r^{1-v}(v-1)^{-1} - \tfrac{1}{2}r^{-v} - v\int_r^{\infty}\psi(x)x^{-v-1}dx = r^{1-v}(v-1)^{-1} - \tfrac{1}{2}r^{-v} - v(v+1)\int_r^{\infty}\psi_1(x)x^{-v-2}dx$$

$$= r^{1-v}(v-1)^{-1} - \tfrac{1}{2}r^{-v} + O(|v|^2 r^{-\mathrm{Re}\, v-1}),$$

since $\psi_1(x) \ll 1$ uniformly in x. Hence

$$f(u,v) = (v-1)^{-1}\sum_{r=1}^{\infty} r^{1-u-v} - \tfrac{1}{2}\sum_{r=1}^{\infty} r^{-u-v} + O(|v|^2\sum_{r=1}^{\infty} r^{-\mathrm{Re}\, u-\mathrm{Re}\, v-1}),$$

and therefore

$$f(u,v) - (v-1)^{-1}\zeta(u+v-1) + \tfrac{1}{2}\zeta(u+v)$$

is analytic for $\mathrm{Re}(u+v) > 0$. Thus (11.15) holds by analytic continuation when u and v both lie in the critical strip, apart from the poles at $v = 1$, $u+v = 1$, $u+v = 2$.

We consider next the case $\mathrm{Re}\, u < 0$, $\mathrm{Re}(u+v) > 2$. Using the Poisson summation formula (1.23) we obtain

$$(11.17)\qquad \sum_{r=0}^{\infty} r^{-u}(r+s)^{-v} = \int_0^\infty x^{-u}(x+s)^{-v}dx + 2\sum_{m=1}^{\infty}\int_0^\infty x^{-u}(x+s)^{-v}\cos(2\pi mx)dx =$$

$$s^{1-u-v}\Big(\int_0^\infty y^{-u}(1+y)^{-v}dy + 2\sum_{m=1}^{\infty}\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi msy)dy\Big),$$

after the change of variable $x = sy$. Summing over s and using (1.29) we have

$$g(u,v) = f(u,v) - \Gamma(u+v-1)\Gamma(1-u)\Gamma^{-1}(v)\zeta(u+v-1) =$$

$$(11.18)\qquad 2\sum_{s=1}^{\infty} s^{1-u-v}\sum_{m=1}^{\infty}\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi msy)dy.$$

To investigate the convergence of the last expression we note that for $\mathrm{Re}\,u < 1$, $\mathrm{Re}(u+v) > 0$, $n \geq 1$

$$2\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi ny)dy = n^{u-1}\int_0^\infty y^{-u}(1+y/n)^{-v}(e(y)+e(-y))dy =$$

$$(11.19)\qquad n^{u-1}\int_0^{i\infty} y^{-u}(1+y/n)^{-v}e(y)dy + n^{u-1}\int_0^{-i\infty} y^{-u}(1+y/n)^{-v}e(-y)dy \ll \frac{n^{\mathrm{Re}\,u-1}}{|u-1|}$$

uniformly for bounded u and v, which follows after integrating by parts. Thus the double series in (11.18) is absolutely convergent for $\mathrm{Re}\,u < 0, \mathrm{Re}\,v > 1$, $\mathrm{Re}(u+v) > 0$, by comparison with $\sum_{s=1}^{\infty}|s|^{-v}\sum_{m=1}^{\infty}|m^{u-1}|$, and represents an analytic function of both variables in this region. Hence (11.18) holds throughout this region and grouping terms with $ms = n$ together we have

$$(11.20)\qquad g(u,v) = 2\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi ny)dy,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the sum of the k-th powers of divisors of n, so that $\sigma_0(n) = d(n)$. Therefore if $g(u,v)$ is the analytic continuation of the function given by (11.18), then for $0 < \mathrm{Re}\,u < 1$, $0 < \mathrm{Re}\,v < 1$, $u + v \neq 1$ we have

$$(11.21)\qquad \zeta(u)\zeta(v) = \zeta(u+v) + \zeta(u+v-1)\Gamma(u+v-1)\Big(\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\Big) + g(u,v) + g(v,u).$$

It is however the exceptional case $u + v = 1$, in which we are interested. Here we may use the fact that $g(u,v)$ is continuous and write $u + v = 1 + \delta$, $0 < |\delta| < 1/2$, with the aim of letting $\delta \to 0$. Then the first terms on the right-hand side of (11.21) become

$$\zeta(1+\delta) + \zeta(\delta)\Gamma(\delta)\left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)} + \frac{\Gamma(u-\delta)}{\Gamma(u)}\right) =$$

$$\zeta(1+\delta) + \zeta(1-\delta)(2\pi)^{\delta}(2\cos\pi\delta/2)^{-1}\left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)} + \frac{\Gamma(u-\delta)}{\Gamma(u)}\right) =$$

$$\delta^{-1} + \gamma + (\gamma - \delta^{-1})\left(\frac{1}{2} + \frac{\delta}{2}\log 2\pi\right)\left(1 - \frac{\Gamma'(1-u)}{\Gamma(1-u)}\delta + 1 - \delta\frac{\Gamma'(u)}{\Gamma(u)}\right) + O(|\delta|) =$$

$$\frac{1}{2}\left(\frac{\Gamma'(1-u)}{\Gamma(1-u)} + \frac{\Gamma'(u)}{\Gamma(u)}\right) + 2\gamma - \log 2\pi + O(|\delta|),$$

where we used Taylor's formula for the gamma-function terms, the functional equation for the zeta-function and

$$\zeta(s) = (s-1)^{-1} + \gamma + O(|s-1|).$$

Hence letting $\delta \to 0$ we have, for $0 < \operatorname{Re} u < 1$,

$$\zeta(u)\zeta(1-u) = \frac{1}{2}\left(\frac{\Gamma'(1-u)}{\Gamma(1-u)} + \frac{\Gamma'(u)}{\Gamma(u)}\right) + 2\gamma - \log 2\pi + g(u,1-u) + g(1-u,u), \tag{11.22}$$

with a view to the eventual application $u = 1/2 + it$ in mind. Reasoning as in (11.19) we have, for $\operatorname{Re} u < 0$,

$$g(u,1-u) = 2\sum_{n=1}^{\infty} d(n)\int_0^{\infty} y^{-u}(1+y)^{u-1}\cos(2\pi ny)\,dy, \tag{11.23}$$

and so what we need is an analytic continuation of (11.23) valid when $\operatorname{Re} u = 1/2$. At this point of the proof the Voronoï formula for $\Delta(x)$ comes into play, since it is a powerful tool which will provide the desired analytic continuation and enable us to integrate (11.22) over t when $u = 1/2 + it$ (t real), thus giving the expression $2i\int_0^T |\zeta(1/2+it)|^2 dt$ on the left-hand side of (11.22). Using the Voronoï formula (3.1) and the asymptotic formulas (3.12) and (3.13) we have, when x is not an integer,

$$\Delta(x) = (\pi 2^{1/2})^{-1}\sum_{n=1}^{\infty} d(n)n^{-3/4}\left(\cos(4\pi\sqrt{nx}-\pi/4) - 3(32\pi\sqrt{nx})^{-1}\sin(4\pi\sqrt{nx}-\pi/4)\right) + O(x^{-3/4}), \tag{11.24}$$

and the series is boundedly convergent in any finite x-interval.

Let now N be a positive integer, and let

$$h(u,x) = 2\int_0^{\infty} y^{-u}(1+y)^{u-1}\cos(2\pi xy)\,dy. \tag{11.25}$$

Then we have with $D(x) = \sum_{n\le x} d(n)$

$$\sum_{n>N} d(n)h(u,n) = \int_{N+1/2}^{\infty} h(u,x)dD(x) = \int_{N+1/2}^{\infty} (\log x + 2\gamma)h(u,x)dx + \int_{N+1/2}^{\infty} h(u,x)d\Delta(x) =$$

$$- \Delta(N+1/2)h(u,N+1/2) + \int_{N+1/2}^{\infty} (\log x+2\gamma)h(u,x)dx - \int_{N+1/2}^{\infty} \Delta(x)\frac{\partial h(u,x)}{\partial x}dx.$$

Hence (11.23) becomes

$$g(u,1-u) = \sum_{n\le N} h(u,n)d(n) - \Delta(N+1/2)h(u,N+1/2) + \int_{N+1/2}^{\infty} (\log x + 2\gamma)h(u,x)dx -$$

(11.26)

$$- \int_{N+1/2}^{\infty} \Delta(x)\frac{\partial h(u,x)}{\partial x}dx = g_1(u) - g_2(u) + g_3(u) - g_4(u),$$

say. Here $g_1(u)$ and $g_2(u)$ are analytic functions of u in the region $\operatorname{Re} u < 1$, since the right-hand side of (11.25) is analytic in this region. Consider next $g_4(u)$. We have

$$h(u,x) = \int_0^{i\infty} y^{-u}(1 + y)^{u-1}e(xy)dy + \int_0^{-i\infty} y^{-u}(1 + y)^{u-1}e(-xy)dy,$$

$$\frac{\partial h(u,x)}{\partial x} = 2\pi i \int_0^{i\infty} y^{1-u}(1+y)^{u-1}e(xy)dy - 2\pi i \int_0^{-i\infty} y^{1-u}(1+y)^{u-1}e(-xy)dy =$$

$$2\pi i x^{u-2}\Big(\int_0^{i\infty} y^{1-u}(1 + y/x)^{u-1}e(y)dy - \int_0^{-i\infty} y^{1-u}(1 + y/x)^{u-1}e(-y)dy\Big) \ll x^{\operatorname{Re}u-2}$$

for $\operatorname{Re} u \le 1$ and bounded u. Using only the estimate $\Delta(x) \ll x^{1/3+\varepsilon}$ it is seen that the integral defining $g_4(u)$ is an analytic function of u at any rate when $\operatorname{Re} u < 2/3$.

It remains to consider $g_3(u)$. Let for brevity $X = N + 1/2$. Then

$$(11.27)\quad g_3(u) = \int_X^{\infty}(\log x+2\gamma)\Big(\int_0^{i\infty} y^{-u}(1+y)^{u-1}e(xy)dy + \int_0^{-i\infty} y^{-u}(1+y)^{u-1}e(-xy)dy\Big).$$

For $\operatorname{Re} u < 0$ an integration by parts shows that the first two integrals in (11.27) are equal to

$$-(2\pi i)^{-1}(\log X+2\gamma) \int_0^{i\infty} y^{-u-1}(1+y)^{u-1}e(Xy)dy - (2\pi i)^{-1}\int_X^{\infty} dx \int_0^{i\infty} y^{-u-1}(x+y)^{u-1}e(xy)dy =$$

$$-(2\pi i)^{-1}(\log X+2\gamma)\int_0^{\infty} y^{-u-1}(1+y)^{u-1}e(Xy)dy + (2\pi i u)^{-1}\int_0^{i\infty} y^{-u-1}(X + y)^{u}e(y)dy.$$

In the last integral above the line of integration may be taken as $[0,\infty)$ and the variable y replaced by $y = Xz$. The other two integrals in (11.27) are treated similarly, and the results may be combined to produce

$$(11.28)\qquad g_3(u) = -\pi^{-1}(\log X + 2\gamma)\int_0^\infty y^{-u-1}(1+y)^{u-1}\sin(2\pi Xy)\,dy + (\pi u)^{-1}\int_0^\infty y^{-u-1}(1+y)^{u}\sin(2\pi Xy)\,dy.$$

Noting that the integrals in (11.28) are uniformly convergent when $\mathrm{Re}\,u \le 1-\varepsilon$, it follows that (11.28) provides us with an analytic continuation which is valid when $\mathrm{Re}\,u = 1/2$, and thus we may proceed to integrate (11.22). When $u = 1/2 + it$ we have $\zeta(u)\zeta(1-u) = |\zeta(1/2+it)|^2$, so that the integration of (11.22) gives

$$2iI(T) = \int_{1/2-iT}^{1/2+iT}\zeta(u)\zeta(1-u)\,du = \tfrac{1}{2}(-d\log\Gamma(1-u) + d\log\Gamma(u))\Big|_{1/2-iT}^{1/2+iT} + 2iT(2\gamma - \log 2\pi)$$

$$+\int_{1/2-iT}^{1/2+iT}(g(u,1-u) + g(1-u,u))\,du = \log\frac{\Gamma(1/2+iT)}{\Gamma(1/2-iT)} + 2iT(2\gamma - \log 2\pi) + {}$$

$$+\,2\int_{1/2-iT}^{1/2+iT} g(u,1-u)\,du.$$

Using Stirling's formula in the form given by (1.31) this becomes

$$(11.29)\qquad I(T) = T\log(T/2\pi) + (2\gamma-1)T - i\int_{1/2-iT}^{1/2+iT} g(u,1-u)\,du + O(1) =$$

$$= T\log(T/2\pi) + (2\gamma-1)T + I_1 - I_2 + I_3 - I_4 + O(1),$$

where for $n = 1,2,3,4$

$$(11.30)\qquad I_n = -i\int_{1/2-iT}^{1/2+iT} g_n(u)\,du,$$

so that using (11.26) and (11.28) we have

$$(11.31)\qquad I_1 = 4\sum_{n\le N} d(n)\int_0^\infty \frac{\sin(T\log(1+y)/y)\cos 2\pi ny}{y^{1/2}(1+y)^{1/2}\log(1+y)/y}\,dy,$$

$$(11.32)\qquad I_2 = 4\Delta(x)\int_0^\infty \frac{\sin(T\log(1+y)/y)\cos 2\pi Xy}{y^{1/2}(1+y)^{1/2}\log(1+y)/y}\,dy,$$

$$(11.33)\qquad I_3 = -\frac{2}{\pi}(\log X + 2\gamma)\int_0^\infty \frac{\sin(T\log(1+y)/y)\sin 2\pi Xy}{y^{3/2}(1+y)^{1/2}\log(1+y)/y}\,dy + {}$$

$$+ \ (\pi i)^{-1} \int_0^\infty y^{-1}\sin(2\pi Xy)\,dy \int_{1/2-iT}^{1/2+iT} (1 + y^{-1})^u u^{-1}\,du,$$

and lastly

(11.34) $$I_4 = -i \int_X^\infty \Delta(x)\,dx \int_{1/2-iT}^{1/2+iT} \frac{\partial h(u,x)}{\partial x}\,du,$$

where N is a positive integer, $X = N + 1/2$, and as in the formulation of the theorem we shall restrict N to the range $AT < N < A'T$. A more explicit formula for I_4 may be derived as follows. We have from (11.25)

$$\int_{1/2-iT}^{1/2+iT} \frac{\partial h(u,x)}{\partial x}\,du = 4i\frac{\partial}{\partial x}\left\{\int_0^\infty \frac{\sin(T(\log(1+y)/y)\cos(2\pi xy)}{y^{1/2}(1 + y)^{1/2}\log(1 + y)/y}\,dy\right\} =$$

$$4i\frac{\partial}{\partial x}\left\{\int_0^\infty \frac{\sin(T\log(x+y)/y)\cos(2\pi xy)}{y^{1/2}(x + y)^{1/2}\log(x + y)/y}\,dy\right\} =$$

$$4i\int_0^\infty \frac{\cos 2\pi y}{y^{1/2}(x+y)^{3/2}\log(x+y)/y}\left\{T\cos(T\log(x+y)/y) - \sin(T\log(x+y)/y)\left(\frac{1}{2} + \log^{-1}\frac{x+y}{y}\right)\right\}dy.$$

Hence replacing y by xy we obtain

(11.35) $$I_4 =$$

$$= 4\int_X^\infty \frac{\Delta(x)}{x}\,dx \int_0^\infty \frac{\cos 2\pi xy}{y^{1/2}(1+y)^{3/2}\log\frac{1+y}{y}}\left\{T\cos\left(T\log\frac{1+y}{y}\right) - \sin\left(T\log\frac{1+y}{y}\right)\left(\frac{1}{2} + \log^{-1}\frac{1+y}{y}\right)\right\}dy.$$

The main difficulty lies now in the evaluation of the integrals which represent I_n. We shall need two lemmas which will follow from Theorem 2.2. These are

Lemma 11.1. Let $\alpha, \beta, \gamma, a, b, k, T$ be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < 1/2$, $a < T/(8\pi k)$, $b \geq T$, $k \geq 1$, $T \geq 1$. Then

(11.36) $$\int_a^b y^{-\alpha}(1 + y)^{-\beta}\left(\log\frac{1+y}{y}\right)^{-\gamma}\exp\left(iT\log\frac{1+y}{y} + 2\pi kiy\right)dy =$$

$$(2k\pi^{1/2})^{-1}T^{1/2}V^{-\gamma}U^{-1/2}(U - 1/2)^{-\alpha}(U + 1/2)^{-\beta}\exp\left(iTV + 2\pi ikU - \pi ik + \frac{\pi i}{4}\right) +$$

$$+ \; O(a^{1-\alpha}T^{-1}) \; + \; O(b^{\gamma-\alpha-\beta}k^{-1}) \; + \; R(T,k)$$

uniformly for $|\alpha - 1| > \varepsilon$, where

$$U = (T/2\pi k + 1/4)^{1/2}, \quad V = 2\operatorname{arsinh}((\pi k/2T)^{1/2}),$$

$$R(T,k) \ll T^{(\gamma-\alpha-\beta)/2-1/4}k^{-(\gamma-\alpha-\beta)/2-5/4}, \qquad \text{for } 1 \le k \le T,$$

$$R(T,k) \ll T^{-1/2-\alpha}k^{\alpha-1}, \qquad \text{for } k \ge T.$$

A similar result holds for the corresponding integral with -k in place of k, except that in that case the explicit term on the right-hand side of (11.36) is to be omitted.

Lemma 11.2. For $AT^{1/2} < a < A'T^{1/2}$, $0 < A < A'$, $\alpha > 0$,

$$(11.37) \int_a^\infty \frac{\exp i\{4\pi x\sqrt{n} - 2T\operatorname{arsinh}(x\sqrt{\pi/2T}) - (2\pi x^2T + \pi^2x^4)^{1/2} + \pi x^2\}}{x^\alpha \operatorname{arsinh}(x\sqrt{\pi/2T})(1/2 + (T/2\pi x^2 + 1/4)^{1/2})(1/4 + T/2\pi x^2)^{1/4}}dx =$$

$$4\pi T^{-1}n^{(\alpha-1)/2}(\log T/2\pi n)^{-1}(T/2\pi - n)^{3/2-\alpha}\exp\{i(T - T\log(T/2\pi n) - 2\pi n + \pi/4)\} +$$

$$+ \; O(T^{-\alpha/2}\min(1, |2\sqrt{n} + a - (a^2+2T/n)^{1/2}|^{-1})) + O(n^{(\alpha-1)/2}(T/2\pi - n)^{1-\alpha}T^{-3/2}),$$

provided that $n \ge 1$, $n < T/2\pi$, $(T/2\pi - n)^2 > na^2$. If the last two restrictions on n are not satisfied, or if $\sqrt{n}$ is replaced by $-\sqrt{n}$, then the main term and the last error term on the right-hand side of (11.37) are to be omitted.

Proof of Lemma 11.1 and Lemma 11.2. To obtain Lemma 11.1 one may apply Theorem 2.2 with

$$\varphi(x) = x^{-\alpha}(1 + x)^{-\beta}\left(\log\frac{1+x}{x}\right)^{-\gamma}, \quad f(x) = (T/2\pi)\log\frac{1+x}{x},$$

$$\Phi(x) = x^{-\alpha}(1 + x)^{\gamma-\beta}, \quad F(x) = T/(1 + x), \quad \mu(x) = x/2.$$

We have

$$f'(x) = -\frac{T}{2\pi x(1 + x)},$$

so that the saddle points of Theorem 2 are the roots of

$$x_0(x_0 + 1) = T/(2\pi k),$$

hence $x_0 = U - 1/2$ in the notation of Lemma 11.1. Thus

$$f_o'' = \frac{T(2x_o + 1)}{2\pi x_o^2(x_o + 1)^2} = 4\pi k^2 U T^{-1},$$

$$\log(1 + 1/x_o) = \log\frac{U + 1/2}{U - 1/2} = \log\frac{(2T/\pi k + 1)^{1/2} + 1}{(2T/\pi k + 1)^{1/2} - 1} =$$

$$\log\frac{(\pi k/2T + 1)^{1/2} + (\pi k/2T)^{1/2}}{(\pi k/2T + 1)^{1/2} - (\pi k/2T)^{1/2}} = \log((\pi k/2T + 1)^{1/2} + (\pi k/2T)^{1/2})^2 =$$

$$= 2\ \mathrm{arsinh}(\pi k/2T)^{1/2}.$$

Hence

$$f_o + kx_o = TV/(2\pi) + k(U - 1/2),$$

and the main term furnished by Theorem 2.2 is

$$\varphi_o(f_o'')^{-1/2} e(f_o + kx_o + 1/8) =$$

$$(U - 1/2)^{-\alpha}(U + 1/2)^{-\beta} V^{-\gamma}(2k)^{-1}(T/\pi)^{1/2} U^{-1/2} \exp\{i(TV + 2\pi kU - \pi k + \pi/4)\}.$$

Consider now the error terms. If $1 \le k \le T$, we have then

$$A \le A(T/k)^{1/2} < x_o < A'(T/k)^{1/2},\ \phi_o \ll x_o^{\gamma-\alpha-\beta},$$

$$\mu_o \ll x_o,\ A(kT)^{1/2} < F_o < A'(kT)^{1/2},$$

and thus for $1 \le k \le T$,

$$\phi_o \mu_o F^{-3/2} \ll T^{\frac{1}{2}(\gamma-\alpha-\beta)-1/4} k^{-(\gamma-\alpha-\beta)/2-5/4},$$

while in case $k \ge T$ we obtain similarly

$$\phi_o \mu_o F^{-3/2} \ll T^{-\alpha-1/2} k^{\alpha-1}.$$

From $f'(x) = -T/2\pi x(1+x)$ we have, for $a < \max(\frac{1}{2}, T/8\pi k)$,

$$f'(x) + k < -ATa^{-1},$$

which gives

$$\phi(a)(f'(a) + k)^{-1} \ll a^{1-\alpha}T^{-1}.$$

Likewise, if $b \ge T$,

$$\phi(b)(f'(b) + k)^{-1} \ll b^{\gamma-\alpha-\beta}k^{-1}.$$

The error-term integral in Theorem 2.2 is

$$\ll \int_a^1 x^{-\alpha} e^{-Akx-AT} dx + \int_1^b x^{\gamma-\alpha-\beta} e^{-Akx-AT/x} dx,$$

and for $|\alpha - 1| > \varepsilon > 0$ the contribution of the above terms clearly does not exceed the order of the error terms given by Lemma 11.1. This establishes Lemma 11.1 for $k \geq 1$, while for $k \leq -1$ the argument differs only in that the the terms in x_o do not occur, since then there are no saddle points in Theorem 2.2.

For the proof of Lemma 11.2 we apply Theorem 2.2 with a,b as limits of integration, where $b > T$, and

$$\varphi(x) = x^{-\alpha}(\operatorname{arsinh}(x\sqrt{\pi/2T}))^{-1}((T/2\pi x^2 + 1/4)^{1/2} + 1/2)^{-1}(T/2\pi x^2 + 1/4)^{-1/4},$$

$$f(x) = \tfrac{1}{2}x^2 - (Tx^2/2\pi + x^4/4)^{1/2} - \tfrac{T}{\pi}\operatorname{arsinh}(x\sqrt{\pi/2T}).$$

We have then

$$f'(x) = x - (x^2 + 2T/\pi)^{1/2}, \quad f''(x) = 1 - x(x^2 + T/2\pi)^{-1/2},$$

so that we may take $\mu(x) = x/2$, $\phi(x) = x^{-\alpha}$, $F(x) = T$. We dispose first of the error terms in a and b. We have

$$\phi(a)(|f'_a + 2\sqrt{n}| + f''^{-1/2}_a)^{-1} \ll T^{-\alpha/2}\min(1, |2\sqrt{n} + a - (a^2 + \tfrac{2}{\pi}T)^{1/2}|^{-1}),$$

and

$$\phi(b)(f'_b + 2\sqrt{n})^{-1} \ll b^{-\alpha}(\sqrt{n} + O(Tb^{-1}))^{-1}$$

which is $o(1)$ for $b \to \infty$. The error-term integral of Theorem 2.2 gives here

$$\ll \int_a^b x^{-\alpha}e^{-Ax\sqrt{n}-AT}dx \ll e^{-A\sqrt{nT}-AT},$$

while

$$\phi_o x_o F_o^{-3/2} \ll x_o^{1-\alpha}T^{-3/2} \ll n^{(\alpha-1)/2}(T/2\pi - n)^{1-\alpha}T^{-3/2},$$

as x_o is given by

$$f'(x_o) + 2\sqrt{n} = 0, \quad x_o = n^{-1/2}(T/2\pi - n).$$

Here if $\sqrt{n} \leq -1$, or $n > T/2\pi$ or $(T/2\pi - n) \leq na^2$ there will be no terms in x_o and the lemma is proved. In other cases we find that

$$f''_o = 2n(T/2\pi + n)^{-1}, \quad \operatorname{arsinh}(x_o\sqrt{T/2\pi}) = \tfrac{1}{2}\log(T/2\pi n),$$

$$(T/2\pi x_o^2 + 1/4)^{1/2} - 1/2 = n(T/2\pi - n)^{-1},$$

$$(T/2\pi x_o^2 + 1/4)^{1/2} + 1/2 = \tfrac{T}{2\pi}(T/2\pi - n)^{-1},$$

etc. and calculating the main term, which is

$$\varphi_o(f''_o)^{-1/2}\exp(2\pi i f_o + 4\pi i x_o\sqrt{n} + \tfrac{1}{4}\pi i),$$

we obtain Lemma 11.2.

Having now at disposal Lemma 11.1 and Lemma 11.2 we proceed to evaluate I_n for $n \le 4$, as given by (11.31) - (11.34). We consider first I_1, taking in Lemma 11.1 $0 < \alpha < 1$, $\alpha + \beta > \gamma$, so that we may let $a \to 0, b \to \infty$. Hence, if $1/2 < \alpha < 3/4$, $1 \le k < AT$ we obtain

$$(11.38) \qquad \int_0^\infty \frac{\sin(T\log(1+y)/y)\cos 2\pi ky}{y^\alpha (1+y)^{1/2}\log(1+y)/y}dy =$$

$$(4k)^{-1}(T/\pi)^{1/2}\cdot\frac{\sin(TV + 2\pi kU - \pi k + \pi/4)}{VU^{1/2}(U-1/2)^\alpha (U+1/2)^{1/2}} + O(T^{-\alpha/2}k^{(\alpha-3)/2}),$$

and since this result holds uniformly in α we may put $\alpha = 1/2$. Taking into account that $\sin(x - \pi k) = (-1)^k \sin x$ we obtain, after substituting (11.38) into (11.31),

$$(11.39) \quad I_1 = 2^{-1/2}\sum_{n\le N}(-1)^n \frac{d(n)}{n^{1/2}}\cdot\frac{\sin(2T\operatorname{arsinh}\sqrt{\pi n/2T} + \sqrt{2\pi nT + \pi^2 n^2} + \pi/4)}{\operatorname{arsinh}\sqrt{\pi n/2T}\cdot(T/2\pi n + 1/4)^{1/2}} + O(T^{-1/4}),$$

taking $AT < N < A'T$. Similarly, from (11.32),

$$(11.40) \qquad I_2 \ll |\Delta(X)|X^{-1/2} \ll T^{-1/6}$$

if we use $\Delta(X) \ll X^{1/3}$.

To deal with I_3 we write (11.33) in the form

$$(11.41) \qquad I_3 = -\frac{2}{\pi}(\log X + 2\gamma)I_{31} + (\pi i)^{-1}I_{32}$$

and consider first I_{31}. We have

$$\int_0^\infty \frac{\sin(T\log(1+y)/y)\sin 2\pi Xy}{y^{3/2}(1+y)^{1/2}\log(1+y)/y}dy = \int_0^{(2X)^{-1}} + \int_{(2X)^{-1}}^\infty \ll T^{-1/2},$$

if the first integral is estimated by the second mean value theorem for integrals as

$$2\pi X\int_0^\xi \frac{\sin(T\log(1+y)/y)}{y(1+y)}\cdot\frac{y^{1/2}(1+y)^{1/2}}{\log(1+y)/y}dy =$$

$$2\pi X\xi^{1/2}(1+\xi)^{1/2}(\log(1+\xi)/\xi)^{-1}\int_\eta^\xi \frac{\sin(T\log(1+y)/y)}{y(1+y)}dy =$$

$$2\pi X\xi^{1/2}(1+\xi)^{1/2}(\log(1+\xi)/\xi)^{-1}\{T^{-1}\cos(T\log(1+y)/y)\}\Big|_\eta^\xi \ll T^{-1/2},$$

where $0 \le \eta \le \xi \le (2X)^{-1}$, and the integral $\int_{(2X)^{-1}}^{\infty}$ is estimated by Lemma 11.1 by treating the main terms on the right-hand side of (11.36) as an error term.

Take next I_{32} and write

$$I_{32} = \int_0^{\infty} y^{-1}\sin(2\pi Xy)\,dy \int_{1/2-iT}^{1/2+iT} \frac{1}{u}\left(\frac{1+y}{y}\right)^u du = \int_0^1 \ldots dy + \int_1^{\infty} \ldots dy = I'_{32} + I''_{32},$$

say. In I'_{32} we have $0 < y \le 1$, hence by the residue theorem

$$\int_{1/2-iT}^{1/2+iT} \left(\frac{1+y}{y}\right)^u u^{-1} du = 2\pi i - \left(\int_{1/2+iT}^{-\infty+iT} + \int_{-\infty-iT}^{1/2-iT}\right)\left(\frac{1+y}{y}\right)^u u^{-1} du = 2\pi i + O(T^{-1}y^{-1/2}),$$

since

$$\int_{1/2+iT}^{-\infty+iT} \left(\frac{1+y}{y}\right)^u u^{-1} du \ll T^{-1}\int_{1/2}^{\infty} \left(\frac{1+y}{y}\right)^t dt \ll T^{-1}y^{-1/2}.$$

Hence

$$I'_{32} = 2\pi i \int_0^1 y^{-1}\sin(2\pi Xy)\,dy + O\left(T^{-1}\int_0^1 |\sin 2\pi Xy|\, y^{-3/2} dy\right) =$$

$$2\pi i\cdot\frac{\pi}{2} + O(X^{-1}) + O\left(T^{-1}\int_0^{X^{-1}} Xy^{-1/2} dy\right) + O\left(T^{-1}\int_{X^{-1}}^{\infty} y^{-3/2} dy\right) =$$

$$\pi^2 i + O(T^{-1/2}).$$

Next, an integration by parts gives

$$I''_{32} = \int_1^{\infty} y^{-1}\sin(2\pi Xy)\,dy \int_{1/2-iT}^{1/2+iT} \left(\frac{1+y}{y}\right)^u u^{-1} du =$$

$$\left[-\frac{\cos 2\pi Xy}{2\pi Xy}\int_{1/2-iT}^{1/2+iT} \left(\frac{1+y}{y}\right)^u u^{-1} du\right]_1^{\infty} - \int_1^{\infty} \frac{\cos 2\pi Xy}{2\pi Xy^2}\int_{1/2-iT}^{1/2+iT} \left(\frac{1+y}{y}\right)^u u^{-1} du -$$

$$-\int_1^{\infty} \frac{\cos 2\pi Xy}{2\pi Xy}\int_{1/2-iT}^{1/2+iT} \left(\frac{1+y}{y}\right)^{u-1} y^{-2} du \ll T^{-1}\log T,$$

since for $y \ge 1$

$$\int_{1/2-iT}^{1/2+iT} \left(\frac{1+y}{y}\right)^u u^{-1} du \ll \int_{1/2-iT}^{1/2+iT} |u^{-1} du| \ll \log T,$$

so that finally

$$I_3 = \pi + O(T^{-1/2}\log T).$$

It remains yet to evaluate I_4, as given by (11.35), which will produce the terms of $\sum_2(T)$ in (11.9) in the final result. We estimate first the inner integrals in (11.35), making $a \to 0$, $b \to \infty$ in Lemma 11.1. We have then in the in the notation of Lemma 11.1, for $k = x > AT$,

$$\int_0^\infty \frac{\cos(T\log(1+y)/y)\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}\log(1+y)/y}dy =$$

$$(4x)^{-1}(T/\pi)^{1/2}\frac{\cos(TV + 2\pi xU - \pi x + \pi/4)}{VU^{1/4}(U - 1/2)^{1/2}(U + 1/2)^{3/2}} + O(T^{-1}x^{-1/2}),$$

and similarly for $r = 1,2$

$$\int_0^\infty \frac{\sin(T\log(1+y)/y)\sin(2\pi xy)}{y^{1/2}(1+y)^{3/2}(\log(1+y)/y)^r}dy = O(T^{1/2}(U-1/2)^{-1/2}x^{-1}) + O(T^{-1}x^{-1/2}) = O(x^{-1/2}).$$

Thus we have

$$I_4 = \int_X^\infty x^{-1}\Delta(x)\left\{\frac{T\cos(2T\operatorname{arsinh}\sqrt{\pi x/2T} + (2\pi xT + \pi^2x^2)^{1/2} - \pi x + \pi/4)}{\sqrt{2x}\operatorname{arsinh}\sqrt{\pi x/2T}\cdot((T/2\pi x + 1/4)^{1/2} + \frac{1}{2})(T/2\pi x + 1/4)^{1/4}} + O(x^{-1/2})\right\}dx.$$

Using $\Delta(x) \ll x^{1/3}$ and changing the variable x to $x^{1/2}$ in the above integral we obtain with the aid of (11.24)

$$(11.42)\quad I_4 = \frac{T}{\pi}\sum_{n=1}^\infty d(n)n^{-3/4}\int_{\sqrt{X}}^\infty \frac{\cos\{2T\operatorname{arsinh}(x\sqrt{\pi/2T})+(2\pi x^2T + \pi^2x^4)^{1/2} - \pi x^2 + \pi/4\}}{x^{3/2}\operatorname{arsinh}(x\sqrt{\pi/2T})((T/2\pi x^2+\frac{1}{4})^{1/2}+\frac{1}{2})(T/2\pi x^2 + \frac{1}{4})^{1/4}}\cdot$$

$$\cdot\{\cos(4\pi x\sqrt{n} - \pi/4) - 3(32\pi x\sqrt{n})^{-1}\sin(4\pi x\sqrt{n} - \pi/4)\}dx + O(T^{-1/6}) =$$

$$= \frac{T}{\pi}\sum_{n=1}^\infty d(n)n^{-3/4}J_n + O(T^{-1/6}),$$

say.

Now it is transparent why a result like Lemma 11.2 was formulated and proved; it is needed to estimate the integral J_n in (11.42). Indeed if $(T/2\pi - n)^2 > nX$, $n < T/2\pi$, that is to say if

$$(11.43)\qquad n < (T/2\pi + X/2) - (X^2/4 + XT/2\pi)^{1/2} = Z,$$

then an application of Lemma 11.2 gives with $\alpha = 3/2, \alpha = 5/2$

$$I_4 = 2\sum_{n<Z} d(n)n^{-1/2}(\log T/2\pi n)^{-1}\cos(T(\log T/2\pi n) - T + \pi/4) +$$

$$+ O(\sum_{n<Z} d(n)n^{-1/2}(T - 2\pi n)^{-1}) + O(T^{-1/2}\sum_{n<Z} d(n)n^{-1/2}(T - 2\pi n)^{-1/2}) +$$

$$+ O(T^{1/4}\sum_{n=1}^\infty d(n)n^{-3/4}\min(1, |2\sqrt{n} + \sqrt{X} - \sqrt{X + 2T/\pi}|^{-1}) + O(T^{-1/6}) =$$

$$I_{41} + I_{42} + I_{43} + I_{44} + O(T^{-1/6}),$$

say. Now I_{41} contributes the main term in $-\sum_2(T)$ in (11.9), while the contribution of the other terms (I_{42} comes from applying Lemma 11.2 to estimate the sine terms in (11.42) with $\alpha = 5/2$) is $\ll \log^2 T$. To see this observe that in view of $AT < X < A'T$ we have

$$Z \ll T, \quad T/2\pi - Z \gg T.$$

Hence

$$I_{42} \ll T^{-1}\sum_{n\le Z} d(n)n^{-1/2} \ll T^{-1/2}\log T,$$

$$I_{43} \ll T^{-1/2}T^{-1/2}\sum_{n\le Z} d(n)n^{-1/2} \ll T^{-1/2}\log T,$$

and it remains yet to deal with I_{44}. Since

$$\left(\tfrac{1}{2}\sqrt{X + 2T/\pi} - \tfrac{1}{2}\sqrt{X}\right)^2 = X/2 + T/2\pi - \sqrt{X^2/4 + XT/2\pi} = Z,$$

we have

$$I_{44} \ll T^{1/4}\sum_{n=1}^{\infty} d(n)n^{-3/4}\min(1, |n^{1/2} - Z^{1/2}|^{-1}) =$$

$$T^{1/4}\Big(\sum_{n\le Z/2} + \sum_{Z/2<n\le Z-Z^{1/2}} + \sum_{Z-Z^{1/2}<n\le Z+Z^{1/2}} + \sum_{Z+Z^{1/2}<n\le 2Z} + \sum_{n>2Z}\Big) =$$

$$T^{1/4}(S_1 + S_2 + S_3 + S_4 + S_5),$$

say. Using partial summation and the crude estimate $\sum_{n\le x} d(n) \sim x\log x$, we obtain

$$S_1 = \sum_{n\le Z/2} d(n)n^{-3/4}(Z^{1/2} - n^{1/2})^{-1} \ll Z^{-1/2}\sum_{n\le Z/2} d(n)n^{-3/4} \ll T^{-1/4}\log T,$$

$$S_2 = \sum_{Z/2<n\le Z-Z^{1/2}} d(n)n^{-3/4}(Z^{1/2} - n^{1/2})^{-1} \ll$$

$$\ll Z^{-1/4}\sum_{Z/2<n\le Z-Z^{1/2}} d(n)(Z - n)^{-1} \ll T^{-1/4}\sum_{Z^{1/2}\le k\le Z/2} d([Z]-k)k^{-1} \ll$$

$$\ll T^{-1/4}\Big(Z\log Z . Z^{-1} + \int_{Z^{1/2}}^{Z/2} t\log t . t^{-2}dt\Big) \ll T^{-1/4}\log^2 T,$$

$$S_3 = \sum_{Z-Z^{1/2}<n\le Z+Z^{1/2}} d(n)n^{-3/4} \ll T^{-1/4}\log T,$$

$$S_4 \ll T^{-1/4}\log^2 T$$

follows analogously as the estimate for S_2, and finally

$$S_5 \ll \sum_{n>2Z} d(n)n^{-3/4}(n^{1/2} - Z^{1/2})^{-1} \ll \sum_{n>2Z} d(n)n^{-5/4} \ll T^{-1/4}\log T.$$

Therefore we obtain

$$I_4 = 2\sum_{n\le Z} d(n)n^{-1/2}(\log T/2\pi n)^{-1}\cos(T(\log T/2\pi n) - T + \pi/4) + O(\log^2 T),$$

and here the limit of summation Z may be replaced by

$$N' = N'(T) = T/2\pi + N/2 - (N^2/4 + NT/2\pi)^{1/2},$$

as in the formulation of Theorem 11.1, with a total error which is certainly $\ll \log^2 T$. This proves then Theorem 11.1 if N is an integer, and if N is not an integer then in (11.4) we replace N by $[N]$ again with an error $\ll \log^2 T$.

§3. Modified Atkinson's formula

Atkinson's formula for E(T), as given by (11.4), has the restriction that N should satisfy $AT < N < A'T$. So far this restriction has not proved to be important in applications, of which (one of) the first was the mean value estimate for $\int_{T-G}^{T+G} |\zeta(1/2+it)|^2 dt$ (Theorem 6.2) which was made by D.R. Heath-Brown [1] and enabled him to obtain the twelfth power moment estimate $M(12) \le 2$. Another application of Atkinson's formula, due to Heath-Brown [2], involves an asymptotic formula for $\int_1^T E^2(t)dt$ and will be presented in §4 of this chapter. For both of these applications the range $AT < N < A'T$ has proved to be quite sufficient, but it seems desirable to have a more flexible form of Atkinson's formula available. M. Jutila's approach [6] of transforming Dirichlet polynomials with the divisor function by the use of Voronoï's formula (used also in our proof of Theorem 6.2) can be also successfully applied here to give

THEOREM 11.2. Let $T^\delta \ll N \ll T^2$ and N' and f(T,n) as in Theorem 11.1. Then

(11.44) $$E(T) = 2^{-1/2}\sum_{n\le N}(-1)^n d(n)n^{-1/4}(\operatorname{arsinh}\sqrt{\pi n/2T})^{-1}(T/2\pi n + 1/4)^{-1/4}\cos(f(T,n))$$
$$- 2\sum_{n\le N'} d(n)n^{-1/2}(\log T/2\pi n)^{-1}\cos(T(\log T/2\pi n) - T + \pi/4) +$$
$$+ O((1 + T^{1/2}N^{-1} + T^{1/4}N^{-1/4})\log^2 T).$$

This formula differs from Atkinson's original formula (11.4) in the error term, which is now a function of N also, but this is compensated by the wide range $T^{\delta} \ll N \ll T^2$, where $\delta > 0$ is arbitrary. If $AT < N < A'T$, then the above error terms reduce to $O(\log^2 T)$, i.e. one obtains exactly Atkinson's (11.4). A proof of (11.44) is given by M. Jutila [6], based on the method of his Theorem 1. To prove (11.44) it suffices to show that if $T^{\delta} \ll N_1 < N_2 \ll T^2$, $N_1 \asymp N_2$, then with $L = \log T$ we have

$$2 \sum_{N'(T,N_2) \le n \le N'(T,N_1)} d(n) n^{-1/2} (\log T/2\pi n)^{-1} \cos(T\log(T/2\pi n) - T + \pi/4) =$$

(11.45)

$$-2^{-1/2} \sum_{N_1 \le n \le N_2} (-1)^n d(n) n^{-1/2} (\operatorname{arsinh} \sqrt{\pi n/2T})^{-1} (T/2\pi n + \tfrac{1}{4})^{-1/4} \cos(f(T,n)) +$$

$$+ O(T^{1/2} N_1^{-1} L^2) + O(L^2 \min((T/N_1)^{1/2}, (T/N_1)^{1/4})) + O(N_1^{1/2} T^{-1} (T^2/N_1)^{\varepsilon}).$$

Here $N'(T,N) = N' = T/2\pi + N/2 - (N^2/4 + NT/2\pi)^{1/2}$, and the idea is to start from (11.4) with $N \asymp T$ and use the Voronoï summation formula to shorten one sum in (11.4) and to lengthen the other. The details of the proof are similar to the proof of Theorem 6.2 and thus will be omitted, but some remarks however will be offered. The case $N_2 \le N_0$ is considered first, where N_0 is fixed and satisfies $T/4\pi \le N'(T,N_0) \le 3T/8\pi$. As in the proof of Theorem 6.2 the summands are multiplied by $e(n) = 1$, which will regulate the distrubution of the saddle points coming from the application of Theorem 2.2. After this, the sum is transformed by the Voronoï formula (3.2), and the integral $\int_{N'(T,N_2)}^{N'(T,N_1)} (\log x + 2\gamma) f(x)dx$ estimated by Lemma 2.2. Since $\exp(iT(\log T/2\pi n)) = n^{-iT}\exp(iT\log T/2\pi)$, the saddle points will be the same as those given by (6.48), except now in the sum on the left-hand side of (11.45) we shall have an extra factor $(\log T/2\pi n)^{-1}$, and as in the proof of Theorem 6.2 we see that

$$\log(T/2\pi x_0) = 2\log((\pi n/2T)^{1/2} + (1 + \pi n/2T)^{1/2}) = 2\operatorname{arsinh}((\pi n/2T)^{1/2}).$$

Therefore calculating

$$\sum \varphi(x_0) f_n''(x_0)^{-1/2} e(f(x_0) + kx_0 + 1/8)$$

by Theorem 2.2 we obtain the right-hand side of (11.45). The error terms in (11.45) are obtained by reasoning analogous to the one given in the proof of Theorem 6.2, when one observes that

$$T/2\pi - N'(T,N_1) \asymp (TN_1)^{1/2}, (\log(T/2\pi x))^{-1} \ll (T/N_1)^{1/2}$$

for $N'(T,N_2) \leq x \leq N'(T,N_1)$. In the case when $T \ll N_1 \ll T^2$ it seems easier to transform the sum on the right-hand side of (11.45) by Voronoï's formula, using actually the averaged sum

$$(11.46)\qquad U^{-1}\int_0^U \sum_{N_1+u\leq n\leq N_2-u} \ldots du$$

similar to the one used in the proof of Theorem 6.2, but with the parameter $U = T^{1/2} + N_1T^{-1}$. The terms arising from saddle points of the sum in (11.46) will be exactly those on the left-hand side of (11.45), and the total contribution of the error terms is given by (11.44). This approach seems less difficult than attempts to adapt Atkinson's original proof of Theorem 11.1, where one encounters considerable difficulties when $N = o(T)$. Furthermore the approach via Voronoï's summation formula may be used to yield an explicit formula for $|\zeta(1/2+it)|^2$ itself, which corresponds to a differentiated form of (11.4) in a certain sense. This result is also given by M. Jutila [6], and it will be stated here as

<u>THEOREM 11.3.</u> Let $t \geq t_o$, $t^{\delta} \ll N \leq t/4\pi$, and let $N' = N'(t,N)$ and $f(t,n)$ be as in Theorem 11.1. Then

$$(11.47)\qquad |\zeta(1/2 + it)|^2 = -2^{1/2}\sum_{n\leq N}(-1)^n d(n)n^{-1/2}(1/4 + t/2\pi n)^{-1/4}\sin(f(t,n)) +$$

$$+ 2\sum_{n\leq N'} d(n)n^{-1/2}\cos(t\log(t/2\pi n) - t - \pi/4) + O(N^{1/4}t^{-1/4}\log^2 t) + O(\log t).$$

The equation (11.47) may be considered as an approximate functional equation for $|\zeta(1/2+it)|^2$, different from the one that follows from (4.11) with $s = 1/2 + it$. However this difference is in some sense not essential, since (4.11) may be used to prove (11.47), as will be explained now. First of all, note that

$$|\zeta(1/2 + it)|^2 = \zeta^2(1/2 + it)\chi^{-1}(1/2 + it),$$

where by (4.4) for $t \geq t_o$

$$(11.48)\qquad \chi(1/2 + it) = (2\pi/t)^{it}e^{it+i\pi/4}(1 + O(t^{-1})),$$

so that (4.11) may be written in the form

$$(11.49)\qquad |\zeta(1/2+it)|^2 = \chi^{-1}(1/2+it)\sum_{n\leq N'} d(n)n^{-1/2-it} + \chi(1/2+it)\sum_{n\leq N'} d(n)n^{-1/2+it} +$$

$$+\ \chi(1/2+it)\sum_{N'<n\le t^2/4\pi^2N'} d(n)n^{-1/2+it}\ +\ O(\log t).$$

Now using (11.48) we have

$$\chi(1/2+it)\sum_{n\le N'} d(n)n^{-1/2+it} + \chi^{-1}(1/2+it)\sum_{n\le N'} d(n)n^{-1/2-it} =$$

$$2\mathrm{Re}\Big\{\exp(it\log(t/2\pi)-it-i\pi/4)\sum_{n\le N'} d(n)n^{-1/2-it}\Big\} + O(N'^{1/2}t^{-1}\log t) =$$

$$2\sum_{n\le N'} d(n)n^{-1/2}\cos(t\log(t/2\pi n)-t-\pi/4) + O(N'^{1/2}t^{-1}\log t).$$

Here the error term is trivially dominated by the error terms in (11.47), and so it is seen that (11.47) reduces to the proof of

(11.50)
$$\sum_{N'\le n\le t^2/4\pi^2N'} d(n)n^{-1/2+it} =$$

$$-2^{1/2}\exp(it\log(t/2\pi)-it-i\pi/4)\sum_{n\le N}(-1)^n d(n)n^{-1/2}\left(\frac{1}{4}+t/2\pi n\right)^{-1/4}\sin(f(t,n)) +$$

$$+\ O(N^{1/4}t^{-1/4}\log^2 t)\ +\ O(\log t).$$

This is again achieved via the Voronoï summation formula (3.2) and the use of the proof of Theorem 6.2. The terms of the sum on the left-hand side of (11.50) are again multiplied by $e(n) = 1$, and an averaged form of the sum, as in (11.46), is considered. The series which appears in Voronoï's formula is split into two parts at $N(1+\varepsilon)$. The terms with $n > N(1+\varepsilon)$ will have no saddle points in view of the range of summation, which is $N' < n \le t^2/4\pi^2N'$, while the terms for $n \le N$ will give rise to saddle points x_0 (given again by (6.48)), which will contribute the main terms on the right-hand side of (11.50). The error terms in (11.47) are small for $N \ll T$, and thus this formula can be also used for the derivation of a variant of Theorem 6.2, and then also for higher power moments of the zeta-function. The proof of (11.47) is notably simpler than the proof of Atkinson's formula (11.4), and (11.47) can be also looked at from another viewpoint in the light of Atkinson's formula. Namely starting from (11.22) we have

(11.51) $$|\zeta(1/2+it)|^2 = 2\mathrm{Re}\ g(u,1-u) + O(\log t),\ u = 1/2+it,$$

where $g(u,1-u)$ is defined by (11.23). Using Voronoï's formula we have

(11.52) $$g(u,1-u) = 2\sum_{n\le N} d(n)\int_0^\infty y^{-u}(1+y)^{u-1}\cos(2\pi ny)\,dy\ +$$

$$+ \int_N^\infty (\log x + 2\gamma) h(u,x)dx + \sum_{n=1}^\infty d(n) \int_N^\infty h(u,x)\alpha(nx)dx,$$

where $\alpha(nx)$ is given by (3.15) and $h(u,x)$ by (11.25). A direct application of Theorem 2.2 gives for $1 \ll N \ll t^2$

$$4\mathrm{Re} \sum_{n\le N} d(n)\int_0^\infty y^{-u}(1+y)^{u-1}\cos(2\pi ny)dy = \tag{11.53}$$

$$-2^{1/2}\sum_{n\le N}(-1)^n d(n)n^{-1/2}\left(\frac{1}{4} + t/2\pi n\right)^{-1/4}\sin(f(t,n)) +$$

$$+ O(N^{1/4}t^{-3/4}\log t) + O(N^{1/2}t^{-1}) + O(\log t),$$

so that combining (11.52) and (11.53) we obtain the main term on the right-hand side of (11.47). However difficulties arise with this approach when one tries to estimate the series on the right-hand side of (11.52), and therefore the first proof of (11.47) seems preferable.

§4. The mean square of E(t)

Let as before

$$E(T) = \int_0^T |\zeta(1/2 + it)|^2 dt - T\log(T/2\pi) - (2\gamma - 1)T.$$

Atkinson's formula (11.4) for $E(T)$ provides the means for obtaining a mean square estimate for $E(t)$ which is analogous to Theorem 10.5. The method of proof, due to D.R. Heath-Brown [2], is similar in nature to Cramér's proof of (10.29) and the result is contained in

THEOREM 11.4.

$$\int_2^T E^2(t)dt = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^\infty d^2(n)n^{-3/2}.T^{3/2} + O(T^{5/4}\log^2 T). \tag{11.54}$$

Proof of Theorem 11.4. It will be sufficient to prove

$$\int_T^{2T} E^2(t)dt = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^\infty d^2(n)n^{-3/2}((2T)^{3/2} - T^{3/2}) + O(T^{5/4}\log^2 T), \tag{11.55}$$

and then to replace T by $T/2, T/2^2$, etc. and to sum all the results. We use Atkinson's formula in the form

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + R(T), \tag{11.56}$$

where $\sum_1(T)$ and $\sum_2(T)$ are given by (1.7) and (1.9), and $R(T) \ll \log^2 T$. Then

$$(11.57)\quad \int_T^{2T} E^2(t)dt = \int_T^{2T} \sum_1^2(t)dt + 2\int_T^{2T} \sum_1(t)(\sum_2(t) + R(t))dt + \\ + \int_T^{2T} (\sum_2(t) + R(t))^2 dt.$$

The main term on the right-hand side of (11.55) will come from the first integral on the right-hand side of (11.57). We choose $N = T$ in Atkinson's formula and proceed to show that

$$(11.58)\quad \int_T^{2T} \sum_1^2(t)dt = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}((2T)^{3/2} - T^{3/2}) + O(T^{1+\epsilon}).$$

To demonstrate this we merely square out $\sum_1(t)$ and integrate term by term, estimating the non-diagonal terms (i.e. those for which $m \neq n$) by the following

Lemma 11.1. Let $g_j(t)$, $(1 \le j \le k)$ and $f(t)$ be continuous, monotonic real-valued functions on $[a,b]$ and let $f(t)$ have a continuous, monotonic derivative on $[a,b]$. If $|g_j(t)| \le M_j$, $(1 \le j \le k)$, $|f'(t)| \ge M_0^{-1}$ on $[a,b]$, then

$$(11.59)\quad \left| \int_a^b \prod_{j=1}^{k} g_j(t)\exp(if(t))dt \right| \le 2^{k+3}\prod_{j=0}^{k} M_j.$$

Proof of Lemma 11.1. The lemma is a straightforward generalization of Lemma 2.1. Recall that if $F(x), G(x)$ are real-valued on $[a,b]$ and $F(x)$ is monotonic, then the second mean-value theorem for integrals states that

$$(11.60)\quad \int_a^b F(x)G(x)dx = F(a)\int_a^{\xi} G(x)dx + F(b)\int_{\xi}^{b} G(x)dx$$

for some $a \le \xi \le b$. Applying (11.60) k times to the real and imaginary part of the integral in (11.59) we obtain

$$\left| \int_a^b \prod_{j=1}^{k} g_j(t)\exp(if(t))dt \right| \le$$

$$2^k\prod_{j=1}^{k} M_j \left(\max_{a \le a_0 < b_0 \le b} \left| \int_{a_0}^{b_0} \cos f(t)\cdot dt \right| + \max_{a \le a_1 < b_1 \le b} \left| \int_{a_1}^{b_1} \sin f(t)\cdot dt \right| \right) \le$$

$$2^{k+1}\prod_{j=0}^{k} M_j \left(\max_{a \le \alpha_0 < \beta_0 \le b} \left| \int_{\alpha_0}^{\beta_0} d\sin f(t) \right| + \max_{a \le \alpha_1 < \beta_1 \le b} \left| \int_{\alpha_1}^{\beta_1} d\cos f(t) \right| \right) \le 2^{k+3}\prod_{j=0}^{k} M_j.$$

Now we return to the proof of Theorem 11.4, noting that the terms of $\Sigma_1^2(t)$ are of the form

$$\frac{1}{4}(-1)^{m+n}d(n)d(m)(mn)^{-1/2}g(t)\cos f(t),$$

where with $f(T,n)$ given by (11.5) we have

$$f(t) = f(t,n) \mp f(t,m),$$

$$g(t) = g_1(t)g_2(t)g_3(t)g_4(t),$$

$$g_1(t) = (t/2\pi n + 1/4)^{-1/4},\ g_2(t) = (t/2\pi m + 1/4)^{-1/4},$$

$$g_3(t) = (\operatorname{arsinh}\sqrt{\pi n/2t})^{-1},\ g_4(t) = (\operatorname{arsinh}\sqrt{\pi m/2t})^{-1}.$$

The contribution of the terms with $m \neq n$ is estimated by Lemma 11.1, where we take $M_1 \ll (n/T)^{1/4}$, $M_2 \ll (m/T)^{1/4}$, $M_3 \ll (T/n)^{1/2}$, $M_4 \ll (T/m)^{1/2}$. Also, since

$$f'(t,n) = 2\operatorname{arsinh}\sqrt{\pi n/2t}$$

we may take

$$M_o \ll T^{1/2}|n^{1/2} \mp m^{1/2}|^{-1}.$$

Thus the contribution of the non-diagonal terms is

$$\ll T\sum_{m\neq n\leq T} d(m)d(n)(mn)^{-3/4}|n^{1/2} - m^{1/2}|^{-1} + T\sum_{n=1}^{\infty}d^2(n)n^{-2} \ll T^{1+\epsilon}$$

by repeating the estimate of (10.34), where the second sum above comes from those terms for which $m = n$ but $f(t) \neq 0$.

The contribution of the diagonal terms $m = n$ to the left-hand side of (11.58) is

$$\frac{1}{4}\sum_{n\leq N} d^2(n)n^{-1}\int_T^{2T} g(t)dt,$$

where we recall that $N = T$. For $|x| < 1$ we have $(\operatorname{arsinh}x)^{-2} = x^{-2} + O(1)$, and for $n \leq N = T$ we thus have

$$g(t) = 2^{3/2}t^{1/2}(\pi n)^{-1/2} + O(n^{1/2}T^{-1/2}),$$

which gives

$$\int_T^{2T}\Sigma_1^2(t)dt = \frac{1}{4}2^{3/2}\pi^{-1/2}\sum_{n\leq T} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}dt + O(\sum_{n\leq T} d^2(n)n^{-1/2}T^{1/2}) + O(T^{1+\epsilon}) =$$

$$= (2\pi)^{-1/2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}\int_T^{2T} t^{1/2}dt + O(T^{3/2}\sum_{n>T} d^2(n)n^{-3/2}) + O(T^{1+\epsilon}) =$$

$$\frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}((2T)^{3/2} - T^{3/2}) + O(T^{1+\varepsilon}).$$

This proves (11.58), and now it remains to consider the mean value of $\sum_2(t)$. We have

$$\int_T^{2T} \sum\nolimits_2^2(t)\,dt \ll T\log^4 T. \tag{11.61}$$

The method of proof will be similar to the proof of (11.58). The choice $N = T$ makes now

$$N' = N'(t) = t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{1/2}$$

dependent on t. However for $T \le t \le 2T$ and $n \le N'(t)$ we have $(\log t/2\pi n)^{-1} \ll 1$, so that the logarithmic factor in $\sum_2(t)$ will cause no trouble. The terms of $\sum_2^2(t)$ are of the form

$$2d(m)d(n)(mn)^{-1/2}g(t)\left\{\cos(t\log(t^2/4\pi^2 mn) - 2t + \pi/2) + \cos(t\log m/n)\right\},$$

where

$$g(t) = (\log t/2\pi m)^{-1}(\log t/2\pi n)^{-1} \ll 1.$$

For each pair m,n we have to integrate over that subinterval of $[T,2T]$ for which $N'(t) \ge \max(m,n)$. Since

$$(t(\log t^2/4\pi^2 mn) - 2t + \pi/2)' \gg |\log m/n|,\quad (t\log m/n)' \gg |\log m/n|,$$

an application of Lemma 11.1 shows that the contribution of the terms $m \ne n$ is

$$\ll \sum_{n\ne m\le T} d(m)d(n)(mn)^{-1/2}|\log m/n|^{-1} \ll$$

$$\ll \sum_{m\ne n\le T} (d^2(m)m^{-1} + d^2(n)n^{-1})|\log m/n|^{-1} \ll \sum_{m\ne n\le T} d^2(n)n^{-1}|\log m/n|^{-1} \ll$$

$$\ll \sum_{n\le T} d^2(n)n^{-1}\sum_{m\le T,\, m\ne n} |\log m/n|^{-1} \ll \sum_{n\le T} d^2(n)n^{-1}(T + n\log T) \ll T\log^4 T,$$

since

$$\sum_{m\le T,\, m\ne n} |\log m/n|^{-1} = \sum_{m\le n-1} (\log n/m)^{-1} + \sum_{n<m\le T} (\log m/n)^{-1} =$$

$$\sum_{r\le n-1} \left(\log\frac{n}{n-r}\right)^{-1} + \sum_{r\le T-n} \left(\log\frac{n+r}{n}\right)^{-1} \ll$$

$$\sum_{r\le n-1} nr^{-1} + \sum_{r\le T-n} (1 + nr^{-1}) \ll T + n\log T.$$

The terms $m = n$ trivially contribute

$$\ll T\sum_{n\le T} d^2(n)n^{-1} \ll T\log^4 T,$$

and therefore (11.61) follows.

The proof of (11.54) is finally obtained by combining (11.57),(11.58), (11.61) and using the Cauchy-Schwarz inequality, since

$$\left(\int_T^{2T} \Sigma_1(t)(\Sigma_2(t) + R(t))dt\right)^2 \ll \int_T^{2T}\Sigma_1^2(t)dt \int_T^{2T} (\Sigma_2^2(t) + R^2(t))dt \ll$$

$$T^{3/2}(T\log^4 T + T\log^4 T) \ll T^{5/2}\log^4 T,$$

$$\int_T^{2T} (\Sigma_2(t) + R(t))^2 dt \le 2\int_T^{2T} (\Sigma_2^2(t) + R^2(t))dt \ll T\log^4 T.$$

This finishes the proof of Theorem 11.4, which gives immediately

Corollary 11.1.

$$E(T) = \Omega(T^{1/4}).$$

This is analogous to $\Delta(x) = \Omega(x^{1/4})$ which follows from Theorem 10.5, but the sharper Ω-results known to hold for $\Delta(x)$ are not known yet to hold for $E(T)$. This should not be surprising, as Atkinson's formula for $E(T)$ was derived with the aid of a formula for $\Delta(x)$, embodied in Voronoї's formula, so that it is natural to expect that problems involving $E(T)$ will be at least as difficult as those involving $\Delta(x)$, and more about the connection between $E(T)$ and $\Delta(x)$ will be found in the next section. Going through the proof of Theorem 11.4 one may observe that the proof enables one to estimate the integral of $E^2(t)$ over a short interval, and that the proof actually gives

Corollary 11.2. For $T^\varepsilon \ll G \le T$ uniformly in G we have

$$\int_{T-G}^{T+G} E^2(t)dt \ll T^\varepsilon(GT^{1/2} + T).$$

This estimate is analogous to (10.51) for $\Delta(x)$, and the main interest in estimates of this sort is that they provide us with a way of estimating $\zeta(1/2 + iT)$, and Corollary 11.2 leads to the classical estimate $\zeta(1/2+iT) \ll T^{1/6+\varepsilon}$. To see this observe that with $L = \log T$ and the notation of (11.3) we have

$$\int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt \ll \int_{-GL}^{GL} \exp(-t^2G^{-2})dI(T+t) =$$

(11.62) $$\int_{-GL}^{GL} \exp(-t^2G^{-2})(\log\frac{T+t}{2\pi} + 2\gamma)dt + O(1) + \int_{-GL}^{GL} E(T+t)tG^{-2}\exp(-t^2G^{-2})dt \ll$$

$$\ll GL + G^{-1}L\left(\int_{T-GL}^{T+GL} E^2(t)dt\right)^{1/2}\left(\int_{T-GL}^{T+GL} dt\right)^{1/2} \ll$$

$$\ll T^{\varepsilon}(G + T^{1/4} + G^{-1/2}T^{1/2}),$$

if we use Corollary 11.2 and the Cauchy-Schwarz inequality. In view of Lemma 6.1 (eq. (6.2) with k = 2) and $\int_{-L^2}^{L^2} \ll \int_{-G}^{G}$ the estimate $\zeta(1/2+iT) \ll T^{1/6+\varepsilon}$ follows from (11.62) with the choice $G = T^{1/3}$. Therefore if we define F(T) by

$$(11.63)\qquad \int_2^T E^2(t)dt = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^{\infty} d^2(n)n^{-3/2}T^{3/2} + F(T),$$

any order estimate $F(T) \ll T^{c+\varepsilon}$, $3/4 \le c \le 5/4$ would give $\zeta(1/2+iT) \ll T^{c/6+\varepsilon}$ by the above method. In analogy with Theorem 10.7 I conjecture that

$$(11.64)\qquad F(T) = \Omega(T^{3/4-\delta})$$

for any $\delta > 0$. By the method of proof of Theorem 10.7 this may be obtained if the truth of the Lindelöf hypothesis is assumed, in which case trivially

$$|E(T_1) - E(T_2)| \le$$

$$\left|\int_{T_1}^{T_2} |\zeta(1/2+it)|^2 dt\right| + \left|T_1(\log T_1/2\pi) + T_1(2\gamma-1) - T_2(\log T_2/2\pi) - T_2(2\gamma-1)\right|$$

$$\ll T^{\varepsilon}|T_1 - T_2|$$

for $T \le T_1, T_2 \le 2T$. Thus it is seen that the method of proof of Theorem 10.7 may be applied and (11.64) follows, but it would be interesting to obtain an unconditional proof of (11.64).

§5. Connection between E(T) and $\Delta(x)$

As mentioned in §1, a comparison between (11.4) and (3.17) shows a similarity between $\sum_1(T)$ and $2\pi\Delta(T/2\pi)$, since apart from the oscillating factor $(-1)^n$ the first $o(T^{1/3})$ terms are asymptotically equal to each other. The influence of $\sum_2(T)$ in Atkinson's formula (11.4) may be usually made small by some averaging processs, so that there is in a certain sense also an analogy between E(T) and $2\pi\Delta(T/2\pi)$, pointed out already by Atkinson [3].

Furthermore, if α_2 and θ_2 are the infima of constants a_2 and c_2 such that $\Delta(x) \ll x^{a_2+\varepsilon}$, $E(T) \ll T^{c_2+\varepsilon}$ for every $\varepsilon > 0$, then one would expect $\alpha_2 = \theta_2 = 1/4$ in view of Theorem 10.5 and Theorem 11.4, and as shown by these theorems the inequalities $\alpha_2 < 1/4$ and $\theta_2 < 1/4$ are impossible. Albeit the equality $\alpha_2 = \theta_2$ is still not known to hold, the best upper bounds $\alpha_2 \leq 35/108$ and $\theta_2 \leq 35/108$ are indeed equal. The bound $\alpha_2 \leq 35/108$ is Theorem 10.1, while it was shown by R. Balasubramanian [1] from (11.12) that the estimation of E(T) may be reduced to the estimation of exponential sums to which the methods of G. Kolesnik used for (10.13) equally apply, and the bound $\theta_2 \leq 35/108$ (given here as Corollary 11.4 by another approach) is a consequence. Following the method of M. Jutila [4] it will be shown that E(T) may be majorized by an expression very similar to the one which is given for $2\pi\Delta(T/2\pi)$ by (3.17) with $(-1)^n$ factor, so that the estimation is reduced to very similar exponential sums, which prompts one to expect that $\alpha_2 = \theta_2$ does hold. Furthermote by Lemma 6.1 it is seen that $\zeta(1/2 + iT) \ll T^{(\theta_2+\varepsilon)/2}$ if $E(T) \ll T^{\theta_2+\varepsilon}$, so that Atkinson's formula shows in fact how the three problems of estimating the order of $\Delta(x)$, $E(T)$ and $\zeta(1/2+iT)$ (and in view of §8 of Chapter 10 one might add P(x) also) may be unified in more or less one problem, with very similar exponential sums appearing in each case. Previously these problems have been treated separately and by different methods, and though we repeat again that $\alpha_2 = \theta_2$ still cannot be proved in general, it is hard to imagine a method for the estimation of exponential sums in question which would yield $\alpha_2 \neq \theta_2$.

Our first task will be technical and consists in introducing a new function $\Delta^*(x)$, which will be similar to $\Delta(x)$ but will contain the oscillating factor $(-1)^n$, thus providing a more exact analogy between E(T) and the error term in the divisor problem. Let us for this purpose consider the function

$$D^*(x) = -D(x) + 2D(2x) - \tfrac{1}{2}D(4x), \tag{11.65}$$

where

$$D(x) = \sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + \Delta(x),$$

so that we may write

$$D^*(x) = x\log x + (2\gamma - 1)x + \Delta^*(x), \tag{11.66}$$

(11.67) $$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x).$$

Now it will turn out that $2\pi\Delta^*(T/2\pi)$ is the "right" analogue of $\Sigma_1(T)$ in Atkinson's formula, since for $N \ll x$ we have

(11.68) $$\Delta^*(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le N}(-1)^n d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}).$$

To see that (11.68) holds use (3.17) with $N \ll x$, viz.

$$\Delta(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2})$$

with $x, 2x, 4x$ and $N, N/2, N/4$ respectively. From (11.67) we have then

(11.69) $$\pi\sqrt{2}x^{-1/4}\Delta^*(x) = -\sum_{n\le N} d(n)n^{-3/4}\cos(4\pi\sqrt{nx} - \pi/4) +$$

$$+ 2^2\sum_{2k\le N} d(k)(2k)^{-3/4}\cos(4\pi\sqrt{2kx} - \pi/4) -$$

$$- 2\sum_{4m\le N} d(m)(4m)^{-3/4}\cos(4\pi\sqrt{4mx} - \pi/4) + O(x^{1/4+\varepsilon}N^{-1/2}).$$

The sums on the right-hand side of (11.69) will give one sum

$$\sum_{r\le N} f(r)r^{-3/4}\cos(4\pi\sqrt{rx} - \pi/4)$$

over natural numbers r, and it remains to consider $f(r)$. If r is odd, then obviously $f(r) = -d(r) = (-1)^r d(r)$, since $2k$ and $4m$ are even. If $r = 2s$, but s is odd, then $d(2s) = 2d(s)$ and so $f(r)$ comes from the first two sums on the right-hand side of (11.69) and equals $f(r) = -d(r) + 2d(r) = (-1)^r d(r)$. Finally if $r = 4q$, observe that from $d(2^a) = a + 1$ we always have $d(4q) = 2d(2q) - d(q)$, so that in this case

$$f(r) = -d(4q) + 2^2 d(2q) - 2d(q) = d(4q) = (-1)^r d(r),$$

and thus (11.68) follows from (11.69).

Next we need an averaged expression for $E(T)$. This will be accomplished by integrating $E(T)$ over very short intervals, the precise meaning of "very short" being given below. Because of the square roots in the expression for $E(T)$ it will be technically more convenient to work with the function

(11.70) $$E_0(x) = E(x^2)$$

than with $E(x)$ directly, and with this in mind we define the averaged integral

$$(11.71) \qquad E_1(x) = G^{-1}\int_{-H}^{H} E_o(x+u)e^{-u^2G^{-2}}du.$$

Here $H = GL = G\log T$, $T^{-a} \le G \le T^{-b}$ for some $1/2 > a > b > 0$. The estimate that we need is contained in

<u>Lemma 11.2.</u> For $\frac{1}{2}T^{1/2} \le x \le \frac{3}{2}T^{1/2}$, $M = G^{-2}L^2$ we have

$$(11.72) \qquad E_1(x) = (2\pi x^2)^{1/4}\sum_{n\le M}(-1)^n d(n)n^{-3/4}e(x^2,n)r(x,n)\cos(f(x^2,n)) + O(T^{\varepsilon}).$$

Here

$$(11.73) \qquad r(x,n) = \exp(-4G^2(x\,\mathrm{arsinh}(\sqrt{\pi n/2}\,x^{-1}))^2),$$

and the expressions for e and f are given by Atkinson's formula, i.e.

$$(11.74) \qquad e(x,n) = (1+\pi n/2x)^{-1/4}(\sqrt{2x/\pi n}\,\mathrm{arsinh}(\sqrt{\pi n/2x}))^{-1} = 1 - \frac{\pi n}{24x} + O(n^2x^{-2}),$$

$$(11.75) \qquad f(x,n) = 2x\,\mathrm{arsinh}\sqrt{\pi n/2x} + (\pi^2n^2 + 2\pi nx)^{1/2} - \pi/4 =$$

$$-\pi/4 + (8\pi nx)^{1/2} + O(n^{3/2}x^{-1/2}),$$

where $n \ll x$ in both (11.74) and (11.75).

<u>Proof of Lemma 11.2</u>. Take $N = T$ in Atkinson's formula. By (11.6) we have

$$(11.76) \qquad E_1(x) = \sum_{j=1}^{2} G^{-1}\int_{-H}^{H}\sum\nolimits_j((x+u)^2)e^{-u^2G^{-2}}du + L^2.$$

Consider here first the term with $j = 1$. By (11.7) this is

$$(11.77) \qquad (2/\pi)^{1/4}G^{-1}\int_{-H}^{H}(x+u)^{1/2}\sum_{n\le T}(-1)^n d(n)n^{-3/4}e((x+u)^2,n)\cos(f((x+u)^2,n)e^{-u^2G^{-2}}du.$$

As in Chapter 6 and Chapter 8 we shall use the exponential integral (1.34), namely

$$\int_{-\infty}^{\infty}\exp(At - Bt^2)dt = (\pi/B)^{1/2}\exp(A^2/4B), \quad \mathrm{Re}B > 0.$$

The choice $H = GL$ in (11.71) makes it possible to replace the limits of integration in (11.77) by $(-\infty,\infty)$ with a negligible error. However before doing this we use Taylor's formula to replace $(x+u)^{1/2}$ by $x^{1/2}$ and likewise $e((x+u)^2,n)$ by $e(x^2,n)$ with a total error which is $\ll 1$. Also by Taylor's formula using $f'(t,n) = 2\,\mathrm{arsinh}\sqrt{\pi n/2t}$ we have

$$(11.78) \qquad f((x+u)^2,n) = f(x^2,n) + 4xu\,\mathrm{arsinh}(\sqrt{\pi n/2}\,x^{-1}) + A(n,x)u^2 + O(T^{-1/2}G^3L^3),$$

where $A(n,x) \asymp (n/T)^{3/2}$, since in view of

$$\operatorname{arsinh} z = z - \frac{1}{2}\cdot\frac{z^3}{3} + \frac{1\cdot 3}{2.4}\cdot\frac{z^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{z^7}{7} + \dots, \quad |z| \le 1,$$

we have with $F(x) = f(x^2,n)$ that $F''(x) \asymp (n/T)^{3/2}$, $F^{(3)}(x) \ll n^{3/2}T^{-2}$ holds. Now we substitute (11.78) into (11.77), using $\exp(iy) = 1 + O(|y|)$ for real y, so that the error term in (11.78) makes a total contribution $\ll G^3L^4 \ll 1$. Then we use (1.34), noting that with the abbreviation $B(n,x) = G^{-2} - A(n,x)i$ the expression in (11.77) becomes

$$(11.79)\qquad (2/\pi)^{1/4}x^{1/2}G^{-1}\sum_{n\le T}(-1)^n d(n)n^{-3/4}e(x^2,n)\cdot$$

$$\cdot \operatorname{Re}\left\{e^{if(x^2,n)}(\pi/B(n,x))^{1/2}\exp\left(-\frac{4(x\operatorname{arsinh}(\sqrt{\pi n/2}\,x^{-1}))^2}{B(n,x)}\right)\right\} + O(1).$$

Here the terms with $n > M = G^{-2}L^2$ make a negligible contribution because of the presence of the exponential factor containing $(x\operatorname{arsinh}\ldots)^2$, and if we replace $B(n,x)$ by G^{-2} using Taylor's formula we make a total error which is $\ll T^{-5/4}G^{-3/2}L^5 \ll 1$.

In this fashion the main term in (11.72) is obtained, and to complete the proof of Lemma 11.2 it remains to show that the term with $j = 2$ in (11.76) is $\ll T^{\varepsilon}$. Since $N = T$ was fixed in the definition of $\sum_1(T)$ in Atkinson's formula, then N' in the definition of $\sum_2(T)$ in Atkinson's formula will depend on $(x+u)^2$. However it is convenient to replace $N'((x+u)^2,T)$ by $N'(x^2,T)$. Recalling that

$$N' = N'(x,T) = T/2\pi + x/2 - (x^2/4 + xT/2\pi)^{1/2},$$

we have

$$N'(x^2,T) - N'((x+u)^2,T) \ll T^{1/2}GL.$$

For $n \le N'((x+u)^2,T)$ we have $\log\frac{(x+u)^2}{2\pi n} \gg 1$ and

$$\left(\log\frac{(x+u)^2}{2\pi n}\right)^{-1} = \left(\log\frac{x^2}{2\pi n}\right)^{-1} + O(T^{-1/2}GL).$$

Therefore by Atkinson's formula, for $|u| \le H$,

$$(11.80)\qquad \sum\nolimits_2((x+u)^2) = -2\sum_{n\le N'(x^2,T)} d(n)n^{-1/2}\left(\log\frac{x^2}{2\pi n}\right)^{-1}\cos(g((x+u)^2,n) + O(T^{\varepsilon}G),$$

where from (11.10) we obtain

$$g((x+u)^2,n) = g(x^2,n) + 2x\log(x^2/2\pi n)\cdot u + (\log(x^2/2\pi n) + 2)u^2 + O(G^3L^3T^{-1/2}).$$

We substitute the expression for $g((x+u)^2,n)$ in (11.80) and argue as in the case $j = 1$, using the integral (1.34). The exponential factor, analogous to the one in (11.79) with $(x\,\mathrm{arsinh}\ldots)^2$, will make each term in the sum $\ll T^{-c}$ for any fixed $c > 0$, while the error term in (11.80) will make the contribution $O(T^{\varepsilon})$ in (11.72) so that Lemma 11.2 follows.

Having proved Lemma 11.2 we shall use it to obtain an expression for $E(T)$ analogous to the expression for $2\pi\Delta^*(T/2\pi)$ which follows from (11.68), except that $\cos(4\pi\sqrt{nx} - \pi/4)$ will be replaced by $\cos(f(T,n))$. We suppose that $T/2 \le t_1 \le T \le t_2 \le 2T$, and with $I(T) = \int_0^T |\zeta(1/2+it)|^2 dt$ we have trivially

$$I(t_1) \le I(T) \le I(t_2).$$

This gives easily

$$E(t_1) + O((T - t_1)\log T) \le E(T) \le E(t_2) + O((t_2 - T)\log T) \tag{11.81}$$

by (11.3), and the idea is to integrate (11.81) over a very short interval using Lemma 11.2. We shall consider the first inequality in (11.81) only, since the other one is treated in exactly the same way. Since the relevant range for the order of $E(T)$ is $T^{1/4} \ll E(T) \ll T^{1/3}$, we suppose that Y is a parameter which satisfies $T^{1/4}L^{-1} \le Y \le T^{1/3}L^{-1}$ and let $G = T^{-1/2}YL^{-2}$, so that G clearly satisfies the condition assumed in Lemma 11.2. Letting

$$t_1 = T - Y + 2(T - Y)^{1/2}u + u^2, \quad |u| \le GL,$$

it is seen that with our choice $G = T^{-1/2}YL^{-2}$ we have $t_1 \le T$ as needed in (11.81), and therefore integrating (11.81) we obtain

$$G^{-1}\int_{-GL}^{GL} E(T - Y + 2(T - Y)^{1/2}u + u^2)e^{-u^2G^{-2}}du + O(YL) \le \sqrt{\pi}E(T). \tag{11.82}$$

But the integral in (11.82) is just $E_1((T - Y)^{1/2})$ by (11.70), and thus (11.81) gives in fact

$$E_1((T - Y)^{1/2}) + O(YL) \le \sqrt{\pi}\,E(T) \le E_1((T + Y)^{1/2}) + O(YL).$$

The integrals $E_1((T \pm Y)^{1/2})$ are evaluated by Lemma 11.2, and setting $X = YL$ it is seen that we obtain

THEOREM 11.5. Let $T \le \sigma \le 2T$, $T^{1/4} \le X \le T^{1/3}$. Then uniformly in σ

$$(11.83)\quad E(\sigma) \ll X + T^{1/4} \sup_{|t-\sigma|\le X} \Big| \sum_{n\le TX^{-2}L^8} (-1)^n d(n) n^{-3/4} e(t,n) r(t^{1/2},n)\cos(f(t,n)) \Big|.$$

The value $TX^{-2}L^8$ appears because $M = G^{-2}L^2 = TY^{-2}L^6 = TX^{-2}L^8$, and the presence of the exponential factors in the proof of Lemma 11.2 which come from the application of (11.34) make it possible to obtain the result for $T \le \sigma \le 2T$. Using partial summation we may remove the factors $e(t,n)$ and $r(t^{1/2},n)$ to obtain the analogue of (11.68), which may be stated as

Corollary 11.3. Let $T^{1/4} \le X \le T^{1/3}$ and $M = TX^{-2}L^8$. Then

$$(11.84)\quad E(T) \ll X + T^{1/4} \sup_{|t-T|\le X} \sup_{u\le M} \Big| \sum_{n\le u} (-1)^n d(n) n^{-3/4} \cos(f(t,n)) \Big|.$$

This is a restricted analogue of (11.68), with N corresponding to $M = TX^{-2}L^8$ here. Since for $n \ll t$ we have

$$f(t,n) = -\pi/4 + 4\pi(nt/2\pi)^{1/2} + O(n^{3/2}t^{-1/2}),$$

it is seen that (11.84) corresponds to $2\pi\Delta^*(T/2\pi)$, and so using Kolesnik's method we obtain easily from (11.84) the analogue of Theorem 10.1, namely

Corollary 11.4.

$$(11.85)\quad E(T) \ll T^{35/108+\varepsilon}.$$

The analogy between $E(t)$ and $2\pi\Delta^*(t/2\pi)$ can be pursued even further. From Theorem 11.4 we have

$$(11.86)\quad \int_2^T E^2(t)\,dt = (C_1 + o(1))T^{3/2},$$

while squaring and integrating (11.68) in the way Theorem 10.5 was derived we obtain

$$(11.87)\quad \int_2^T \Delta^{*2}(t)\,dt = (C_2 + o(1))T^{3/2},$$

which shows that the average order of both $|E(t)|$ and $|\Delta^*(t)|$ is $\ll t^{1/4}$. However if we define

$$(11.88)\quad E^*(t) = E(t) - 2\pi\Delta^*(t/2\pi),$$

then it can be shown that the average order of $|E^*(t)|$ is $\ll t^{1/6}\log^{3/2} t$. This follows from

THEOREM 11.6.

$$(11.89)\qquad \int_2^T E^{*2}(t)dt \ll T^{4/3}\log^3 T.$$

Proof of Theorem 11.6. The general idea of the proof is the same one that was used in the proof of Theorem 10.5. It will be sufficient to prove (11.89) for the integral over $[T,2T]$ and we apply Atkinson's formula with $N = T$ in $\Sigma_1(t)$. The quality of the final result in (11.89) is determined by the size of the error term in the expansion for $f(t,n)$ in (11.75), which is small for $n = o(t^{1/3})$. Write

$$(11.90)\qquad \Sigma_1(t) = \Sigma_{11}(t,X) + \Sigma_{12}(t,X),$$

where in Σ_{11} summation is over $n \le X$, and in Σ_{12} over $X < n \le T$. If we set

$$(11.91)\qquad S(t,X) = 2^{1/2}(t/2\pi)^{1/4}\sum_{n\le X}(-1)^n d(n)n^{-3/4}\cos(f(t,n)),$$

then from (11.7) and (11.8) we infer

$$\Sigma_{11}(t,X) - S(t,X) \ll T^{-3/4}\sum_{n\le X} d(n)n^{1/4} \ll T^{-3/4}X^{5/4}\log X \ll \log T$$

with the choice

$$(11.92)\qquad X = T^{1/3}.$$

We use now (11.68) with $N = T$, $x = t/2\pi$ and decompose the sum similarly as the sum in (11.90):

$$(11.93)\qquad \Delta^*(t/2\pi) = \Delta_1^*(t/2\pi,X) + \Delta_2^*(t/2\pi,X) + O(T^\varepsilon).$$

Therefore we obtain

$$\int_2^T E^{*2}(t)dt \ll \int_2^T (S(t,X) - 2\pi\Delta_1^*(t/2\pi,X))^2 dt + \int_T^{2T} \Sigma_{12}^2(t,X)dt +$$

$$+ \int_T^{2T} \Sigma_2^2(t)dt + \int_T^{2T} \Delta_2^{*2}(t/2\pi,X)dt + T^{1+\varepsilon} = \sum_{j=1}^4 I_4 + O(T^{1+\varepsilon}),$$

say. By (11.61) we have $I_3 \ll T\log^4 T$, and likewise the non-diagonal terms (those with $m \ne n$ when the sum is squared) of I_2 contribute $\ll T^{1+\varepsilon}$, while the diagonal terms give trivially

$$\ll T^{3/2}\sum_{n>X} d^2(n)n^{-3/2} \ll T^{4/3}\log^3 T$$

with the choice $X = T^{1/3}$, and the same argument applies to I_4 as well, hence

$$I_2 + I_3 + I_4 \ll T^{4/3}\log^3 T.$$

It remains to estimate I_1. Using $\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$ and defining

$$h_{\mp}(t,n) = \frac{1}{2}(f(t,n) \mp (-\pi/4 + 2(2\pi nt)^{1/2}))$$

we have

$$S(t,X) - 2\pi\Delta_1^*(t/2\pi,X) = -2(2t/\pi)^{1/4}\sum_{n\le X}(-1)^n d(n)n^{-3/4}\sin(h_-(t,n))\sin(h_+(t,n)).$$

Hence

$$I_1 \ll T^{1/2}\sum_{m,n\le X} d(m)d(n)(mn)^{-3/4}\left|\int_T^{2T}\sin(h_-(t,m))\sin(h_-(t,n))\sin(h_+(t,m))\sin(h_+(t,n))dt\right|.$$

As in the proof of Theorem 10.5 and Theorem 11.4 we may estimate the non-diagonal terms $m \ne n$ above by Lemma 2.1 to obtain a total contribution which is $\ll T^{1+\varepsilon}$. As for the diagonal terms, observe that by (11.75)

$$h_-(t,n) \ll n^{3/2}t^{-1/2} \ll n^{3/2}T^{-1/2},$$

and thus using $|\sin x| \le |x|$ for x real we get a contribution which is

$$\ll T^{1/2}\sum_{n\le X} d^2(n)n^{-3/2}\int_T^{2T}\sin^2(h_-(t,n))dt \ll T^{1/2}\sum_{n\le X} d^2(n)n^{3/2} \ll$$

$$\ll T^{1/2}X^{5/2}\log^3 X \ll T^{4/3}\log^3 T,$$

which completes the proof of Theorem 11.6.

§6. Large values and power moments of E(T)

Pursuing further the analogy between $E(T)$ and the divisor problem we present now estimates for power moments of $E(T)$. These estimates are the analogues of Theorem 10.9 and Theorem 10.12, and the result is contained in

THEOREM 11.7.

(11.94) $$\int_2^T |E(t)|^A dt \le T^{(A+4+\varepsilon)/4}, \qquad \text{for } 0 \le A \le 35/4,$$

(11.95) $$\int_2^T |E(t)|^A dt \le T^{(35A+38+\varepsilon)/108}, \qquad \text{for } A \ge 35/4.$$

The proof of Theorem 11.7 is completely analogous to the proof of Theorem 10.9, using (11.85) instead of (10.13) and (11.83) as the analogue of the truncated Voronoĭ formula (3.17) with $TX^{-2}L^8$ corresponding to N. A large values estimate for $E(T)$, namely

$$(11.96) \qquad R \ll T^{\varepsilon}(TV^{-3} + T^{15/4}V^{-12}), \qquad T^{1/4} \ll V \ll T^{1/3},$$

is deduced for $E(T)$ in the same way as (10.54) was derived, and the restriction $T^{1/4} \ll V \ll T^{1/3}$ is not essential, since $V \gg T^{1/3}$ cannot hold because of Corollary 11.4 and for $V \ll T^{1/4}$ one will trivially obtain (11.94) for the corresponding discrete sum. For (11.96) we suppose that $T/2 \le t_1 < \ldots < t_R \le T$ are points which satisfy $|t_r - t_s| \ge CV$ $(r \ne s \le R)$ for some suitable $C > 0$, $T^{1/4} \ll V \ll T^{1/3}$ and $E(t_r) \gg V$ for $r = 1,\ldots,R$. Choosing $X = CV$ we have then from (11.83)

$$(11.97) \qquad R \ll$$

$$\ll T^{1/2+\varepsilon}V^{-2} \max_{M \le CTV^{-2}L^8} \sum_{r \le R} \Big| \sum_{M < n \le 2M} (-1)^n d(n) n^{-3/4} e(t_r', n) r(t_r'^{1/2}, n) \exp(if(t_r', n)) \Big|^2,$$

where t_r' is the point for which the supremum in (11.83) is attained. Considering separately $t'_{4m}, t'_{4m+1}, t'_{4m+2}, t'_{4m+3}$ we may suppose that $|t_r' - t_s'| \ge CV$ when $r \ne s$. From this point the proof of (11.96) is almost identical with the proof of (10.54), since after the application of the Halász-Montgomery inequality (1.36) the functions e and r which appear in (11.97) may be easily removed by partial summation keeping in mind that $r(x,n)$ is monotonic and ≤ 1 and that (11.74) holds. Similarly one has (11.75) for $f(t,n)$, and the theory of exponent pairs that was used in the proof of (10.54) may be equally well applied here, producing (11.96) which yields then Theorem 11.7 with the aid of (11.85).

In analogy with (10.60) it may be noted that the theoretical limit for power moments that (11.96) can give is

$$\int_2^T |E(t)|^{11} dt < T^{15/4+\varepsilon},$$

which would give then

$$(11.98) \qquad \zeta(1/2 + iT) \ll T^{5/32+\varepsilon}.$$

Namely using Lemma 6.1, Hölder's inequality and arguing as in (11.62) we have with $L = \log T$

$$|\zeta(1/2 + iT)|^2 \ll GT^{\varepsilon}\Big(1 + G^{-2}\int_{T-GL}^{T+GL} |E(t)|\,dt\Big) \ll$$

$$GT^{\varepsilon}\Big(1 + G^{-2}\Big(\int_2^{2T} |E(t)|^{11} dt\Big)^{1/11} G^{10/11}\Big) \ll T^{5/16+\varepsilon}$$

for $G = T^{5/16}$.

N O T E S

J.E. Littlewood proved (11.3) in [1] by means of results connected with the approximate functional equation for the zeta-function. E.C. Titchmarsh's book [8] contains a proof of $E(T) \ll T^{1/2+\varepsilon}$ and a proof that $E(T) \ll T^{5/12+\varepsilon}$ has been given by Titchmarsh [4].

The Riemann-Siegel formula used by R. Balasubramanian [1] in his proof of (11.12) is Theorem 4.16 of Titchmarsh [8] with N = 5. The idea of Balasubramanian's proof is to square the expression for $e^{i\theta}\zeta(1/2+it)$ and to estimate the resulting integrals, some of which are technically rather complicated. His paper also contains the result $\gamma_{n+1} - \gamma_n \ll \gamma_n^{1/6+\varepsilon}$, which was discussed in Chapter 8, plus some related results concerning Dirichlet series. Discussing estimates of E(T), Balasubramanian mentions Atkinson's formula (11.4) in his §1 by saying: "In this connection, we can also mention that our result seems to be more useful than that of Atkinson". In view of (11.84) and other applications of Atkinson's formula one could hardly agree with this statement.

In §2 we have followed closely Atkinson's original proof [3] of (11.4), where curiously in l.3 on p. 375 he makes a mistake in sign, obtaining $+2\sum_{n\le N'} d(n)\ldots$ in place of $-2\sum_{n\le N'} d(n)\ldots$. The corrected form of (11.4) was stated by M. Jutila [6] without comment. For technical reasons (to avoid the last term in $\sum_{n\le x}{}' d(n)$ in (3.1)) one takes $X = N + 1/2$, N an integer, from (11.27) onwards, and it is easily seen that this restriction does not affect the final result.

Heath-Brown's derivation [1] of Theorem 6.2 starts from (11.62) in the form

$$\int_{T-G}^{T+G} |\zeta(1/2 + it)|^2 dt \ll GL + \int_{-GL}^{GL} E(T + t)tG^{-2}e^{-t^2G^{-2}}dt,$$

and uses Atkinson's formula. The contribution of $\Sigma_2(T + t)$ to the above integral is small, and the main contribution arises from $\Sigma_1(T + t)$, producing a sum of length $\ll TG^{-2}\log^2 T$. In analyzing the differnce in the proofs of Theorem 6.2 it should be notes that in Heath-Brown's proof one uses first the Voronoï summation formula, which is implicit in Atkinson's formula, and then exponential integrals of the form (11.34), while in our proof one makes first an exponential averaging of the approximate functional equation for $\zeta^2(1/2 + it)$ and then applies the Voronoï summation formula.

It should be perhaps stressed that one of the chief merits of Atkinson's formula or one of its variants like Theorem 11.3 is that the explicit formula in question contains a sum (and not a square or an absolute value etc.) which may be directly integrated termwise as in Lemma 11.2 or in Heath-Brown's proof [1] of Theorem 6.2. Take for example (11.47) and set

$$N = TG^{-2}L^2, \ L = \log T, \ T/2 \le t_r = t \le T, \ T^{1/5+\varepsilon} \le G \le T^{1/3}.$$

Results of §2 of Chapter 7 concerning power moments of the zeta-function may be successfully obtained then from (11.47), where for our range of G we may replace $(1/4 + t_r/2\pi n)^{-1/4}$ by $(t_r/2\pi n)^{-1/4}$ with an error which is $\ll 1$. Namely the basic step in the proof of Theorem 7.1 consists in bounding the sum

$$\sum_{t_r \in A} \int_{t_r-G}^{t_r+G} |\zeta(1/2 + it)|^2 dt \tag{11.99}$$

over some suitable points $t_r \in A \subseteq [T/2, T]$, where $|t_r - t_s| \ge G$ for $t_r \ne t_s \in A$. Averaging (11.47) with the usual exponential factor $\exp(-t^2G^{-2})$ and using (1.34) we infer that the expression in (11.99) is bounded by a constant times

$$G\sum_{t_r \in A} \sum_{n \le TG^{-2}L^2} (-1)^n d(n)(nt_r)^{-1/4} \sin(f(t_r,n)) \exp(-(G \operatorname{arsinh}\sqrt{\pi n/2t_r})^2) +$$

$$+ |A| G\log T = GS + |A| G \log T,$$

say. In Chapter 7 the Halász-Montgomery inequality (1.35) was used to bound in Lemma 7.1 a sum very similar to S above, but the point here is that S can be estimeted directly from first principles with the same effect as if one used (1.35). This is possible since there is no absolute value sign in S, and thus the order of summation can be changed. Since $\exp(-(G\operatorname{arsinh}\sqrt{\pi n/2t_r})^2)$ can be easily removed from S by partial summation as in the proof of Theorem 6.2, it is seen that for $K \le \frac{1}{2}TG^{-2}L^2$ the sum S is in fact majorized by the largest of sums of the type

$$\Big|\sum_{t_r} \sum_{K<n\le 2K} (-1)^n d(n)(nt_r)^{-1/4} \sin(f(t_r,n))\Big| \le$$

$$\Big(\sum_{K<n\le 2K} d^2(n) n^{-1/2}\Big)^{1/2} \Big(\sum_{K<n\le 2K} \Big|\sum_{t_r} t_r^{-1/4} \exp(if(t_r,n))\Big|^2\Big)^{1/2} \ll$$

$$\ll K^{1/4}\log^{3/2}T\Big(\sum_{K<n\le 2K}\ \sum_{t_r,t_s\in A}(t_rt_s)^{-1/4}\exp(if(t_r,n)-if(t_s,n))\Big)^{1/2} \ll$$

$$T^{-1/4}K^{1/4}\log^{3/2}T\Big(|A|K + \sum_{t_r\ne t_s\in A}\Big|\sum_{K<n\le 2K}\exp(if(t_r,n)-if(t_s,n))\Big|\Big)^{1/2}.$$

This exactly corresponds to the use of Lemma 7.1, namely (7.17) in the proof of Theorem 7.1, since

$$\sum_{K<n\le 2K}\exp(if(t_r,n)-if(t_s,n)),\quad t_r\ne t_s,$$

can be estimated either by Lemma 2.5 and Lemma 2.1 or by the theory of exponent pairs as in the proof of Lemma 7.1, and the end result will be therefore the same. The same observation may be made concerning (6.25), where following the proof one obtains the sum on the right-hand side of (6.25) without the absolute value signs with $\exp(if(T,n))$ replaced by $\sin(f(T,n))$. Thus the bounds for the sum in (11.99) may be obtained without the use of (1.35).

The results of §3 are to be found in M. Jutila's work [6], where more general applications of Voronoï type summation formulas are considered.

Theorem 11.4 is due to D.R. Heath-Brown [2]. The result $E(T) = \Omega(T^{1/4})$, stated here as Corollary 11.2, was obtained a little before Heath-Brown by A. Good [1], who used a complicated technique which was not based on Atkinson's formula. A plausible conjecture is that $F(T) \ll T^{3/4+\varepsilon}$, where $F(T)$ is defined by (11.63), and this would lead to the hypothetical $\zeta(1/2 + iT) \ll T^{1/8+\varepsilon}$, a result out of reach at present. The same would follow of course from the conjectural estimate $E(T) \ll T^{1/4+\varepsilon}$, which in view of (11.84) seems to be of the same degree of difficulty as the classical conjecture $\alpha_2 = 1/4$ in the divisor problem. There seems to be no method available at present which would permit one to deduce from the (global) estimate $E(T) \ll T^{c+\varepsilon}$, $1/4 \le c \le 35/108$, anything better than the obvious (local) estimate $\zeta(1/2 + iT) \ll T^{c/2+\varepsilon}$. Proving that such a method does not exist would lead at once to the falsity of the Lindelöf hypothesis.

Using the method of K.S. Gangadharan [1] one ought to be able to prove

$$E(T) = \Omega_{\pm}(T^{1/4}(\log\log T)^{1/4})$$

(in analogy to $\Delta(x) = \Omega_{\pm}(x^{1/4}(\log\log x)^{1/4})$)), and then prove (11.64) also.

The analysis and connection between local and global estimates of $\Delta(x)$ and $\zeta(1/2 + iT)$ has been thoroughly discussed by M. Jutila [4], [5], where the results of §5 may be found. Jutila's analytic proof [4] of (11.68) has been however replaced here by a short elementary argument which uses (3.17). In [5] Jutila proves a more general estimate than Theorem 11.6, namely

$$\int_{T-H}^{T+H} E^{*2}(t)dt \ll HT^{1/3}\log^3 T + T^{1+\varepsilon}, \; 2 \leq H \leq T,$$

but the proof of this more general result is the same as the proof of (11.89). The method of proof of Theorem 10.8 and Theorem 10.9 could be used to furnish an estimate for $\int_1^T E^{*4}(t)dt$, showing that in the mean sense $|E^*(t)|$ is of the order t^c for some c between 1/5 and 1/4, while Theorem 11.6 shows that in mean square $|E^*(t)|$ is of the order $t^{1/6}\log^{3/2} t$.

Interesting conditional results are obtained by M. Jutila [6], and they seem to be the first ones of their kind. For example, if the conjecture that $\alpha_2 = 1/4$ in the divisor problem is true, then Jutila proved

$$\zeta(1/2 + iT) \ll T^{3/20+\varepsilon}, \; E(T) \ll T^{5/16+\varepsilon},$$

and the exponents in the above estimates are better than the best unconditional values 35/216 and 35/108 respectively. These results may be compared with the conditional estimate (11.98), which is the limit of (11.96).

Theorem 11.7 has been given by the author in [6].

REFERENCES

[1] T.M. Apostol, Introduction to analytic number theory, New York - Heidelberg - Berlin, 1976.

[1] F.V. Atkinson, A divisor problem, Quart. J. Math. (Oxford) 12(1941), 193-200.

[2] F.V. Atkinson, The mean value of the zeta-function on the critical line, Proc. London Math. Soc. (2) 47(1941), 174-200.

[3] F.V. Atkinson, The mean value of the Riemann zeta-function, Acta Mathematica 81(1949), 353-376.

[1] R. Balasubramanian, An improvement on a theorem of Titchmarsh on the mean square of $|\zeta(1/2+it)|$, Proc. London Math. Soc. 36(1978), 540-576.

[1] B.C. Berndt, Arithmetical identities and Hecke's functional equation, Proc. Edinburgh Math. Soc. 16(1969), 221-226.

[2] B.C. Berndt, Identities involving the coefficients of a class of Dirichlet series I, Trans. Amer. Math. Soc. 137(1969), 345-359.

[3] B.C. Berndt, Identities involving the coefficients of a class of Dirichlet series V, Trans. Amer. Math. Soc. 160(1971), 139-156.

[4] B.C. Berndt, Identities involving the coefficients of a class of Dirichlet series VII, Trans. Amer. Math. Soc. 201(1975), 247-261.

[1] R. Brent, On the zeros of the Riemann zeta-function in the critical strip, Math. Comp. 33(1979) no. 148, 1361-1372.

[1] K. Chandrasekharan, Arithmetical functions, Berlin-Heidelberg-New York, 1970.

[1] K. Chandrasekharan and R. Narasimhan, Hecke's functional equation and arithmetical identities, Annals of Math. (2)74(1961), 1-23.

[2] K. Chandrasekharan and R. Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Annals of Math. (2)76(1962), 93-136.

[1] Chen Jing-run, On the order of $\zeta(1/2+it)$, Chinese Math.-Acta 6(1965), 463-478.

[2] Chen Jing-run, On the divisor problem for $d_3(n)$, Sci. Sinica 14(1965), 19-29.

[1] Chih Tsung-tao, A divisor problem, Acad. Sinica Science Record 3(1950), 177-182.

[1] J.G. van der Corput, Zahlentheoretische Abschätzungen, Math. Annalen 84(1921), 53-79.
[2] J.G. van der Corput, Verschärfung der Abschätzung beim Teilerproblem, Math. Annalen 87(1922), 39-65.
[3] J.G. van der Corput, Zum Teilerproblem, Math. Annalen 98(1928), 697-716.

[1] K. Corrádi and I. Kátai, A comment on K.S. Gangadharan's paper entitled "Two classical lattice point problems", Magyar Tud. Akad. Mat. Fiz. Oszt. Kozl. 17(1967), 89-97.

[1] H. Cramér, Über zwei Sätze von Herrn G.H. Hardy, Math. Zeit. 15(1922), 200-210.
[2] H. Cramér, On the order of magnitude of the difference between consecutive primes, Acta Arith. 2(1936), 23-46.

[1] J.-M. Deshouillers and H. Iwaniec, Power mean values of the Riemann zeta-function, Mathematika vol. 29 part 2 (1982) p. 202-212.

[1] A.L. Dixon and W.L. Ferrar, Lattice point summation formulae, Quart. J. Math. (Oxford) 2(1931), 31-54.
[2] A.L. Dixon and W.L. Ferrar, On divisor transforms, Quart. J. Math. (Oxford) 3(1932), 43-59.

[1] H.M. Edwards, Riemann's zeta-function, New York-London, 1974.

[1] M. Forti and C. Viola, Density estimates for the zeros of L-functions, Acta Arith. 23(1973), 379-391.
[1] A. Fujii, On the problem of divisors, Acta Arith. 31(1976), 355-360.

[1] R.M. Gabriel, Some results concerning the integrals of moduli of regular functions along certain curves, J. London Math. Soc. 2(1927), 112-117.

[1] K.S. Gangadharan, Two classical lattice point problems, Proc. Cambridge Phil. Soc. 57(1961), 699-721.

[1] D.A. Goldston, Large differences between consecutive prime numbers, Doctoral Thesis, University of California at Berkeley, Berkeley, 1981.

[1] A. Good, Ein Ω-Resultat für das quadratische Mittel der Riemannschen Zetafunktion auf der kritische Linie, Invent. Math. 41(1977), 233-251.

[1] J.L. Hafner, New omega theorems for two classical lattice point problems, Inventiones Math. 63(1981), 181-186.

[2] J.L. Hafner, On the representation of the summatory functions of a class of arithmetical functions, in Analytic Number Theory, LNM 899(1981), 145-165.

[1] G. Halász and P. Turán, On the distribution of roots of Riemann zeta and allied functions I, Journal of Number Theory 1(1969), 121-137.

[1] W. Haneke, Verschärfung der Abschätzung von $\zeta(1/2 + it)$, Acta Arith. 8 (1962-63), 357-430.

[1] G.H. Hardy, On the expression of a number as a sum of two squares, Quart. J. Math. (Oxford) 46(1915), 263-283.

[2] G.H. Hardy, On Dirichlet's divisor problem, Proc. London Math. Soc. (2)15 (1916), 1-25.

[3] G.H. Hardy, The average orders of the functions P(x) and $\Delta(x)$, Proc. London Math. Soc. (2)15(1916), 192-213.

[1] G.H. Hardy and J.E. Littlewood, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, Acta Math. 41(1918), 119-196.

[2] G.H. Hardy and J.E. Littlewood, The approximate functional equation in the theory of the zeta-function, with applications to the divisor problems of Dirichlet and Piltz, Proc. London Math. Soc. (2)21(1922), 39-74.

[3] G.H. Hardy and J.E. Littlewood, The approximate functional equation for $\zeta(s)$ and $\zeta^2(s)$, Proc. London Math. Soc. (2)29(1929), 81-97.

[1] D.R. Heath-Brown, The twelfth power moment of the Riemann zeta-function, Quart. J. Math. (Oxford) 29(1978), 443-462.

[2] D.R. Heath-Brown, The mean value theorem for the Riemann zeta-function, Mathematika 25(1978), 177-184.

[3] D.R. Heath-Brown, The fourth power moment of the Riemann zeta-function, Proc. London Math. Soc. (3) 38(1979), 385-422.

[4] D.R. Heath-Brown, Zero-density estimates for the Riemann zeta-function and Dirichlet L-functions, J. London Math. Soc. 19(1979), 221-232.

[5] D.R. Heath-Brown, Simple zeros of the Riemann zeta-function on the critical line, Bull. London Math. Soc. 11(1979), 17-18.

[6] D.R. Heath-Brown, A large values estimate for Dirichlet polynomials, J. London Math. Soc. 20(1979), 8-18.

[7] D.R. Heath-Brown, Fractional moments of the Riemann zeta-function, J. London Math. Soc. 24(1981), 65-78.

[8] D.R. Heath-Brown, Mean values of the zeta-function and divisor problems, Recent progress in analytic number theory, Symp. Durham 1979 (vol. 1), 1981, 115-119.

[9] D.R. Heath-Brown, Mean value theorems for the Riemann zeta-function, Séminaire Delange-Pisot-Poitou, Paris 1979/1980, Birkhaüser Verlag 1981, 123-134.

[1] D.R. Heath-Brown and H. Iwaniec, On the difference between consecutive primes, Inventiones Math. 55(1979), 49-69.

[1] D.A. Hejhal, A note on the Voronoï summation formula, Monatshefte Math. 87 (1979), 1-14.

[1] M.N. Huxley, The distribution of prime numbers, Oxford, 1972.

[2] M.N. Huxley, On the difference between consecutive primes, Inventiones Math. 15(1972), 155-164.

[3] M.N. Huxley, Large values of Dirichlet polynomials, Acta Arith. 24(1973), 329-346; II ibid. 27(1975), 159-169; III ibid. 26(1975), 435-444.

[1] A.E. Ingham, Mean value theorems in the theory of the Riemann zeta-function, Proc. London Math. Soc. (2) 27(1926), 273-300.

[2] A.E. Ingham, On the estimation of N(σ,T), Quart. J. Math. (Oxford) 11(1940), 291-292.

[1] M.I. Israilov, Coefficients of the Laurent expansion of the Riemann zeta-function (Russian), Dokl. Akad. Nauk UzSSR 12(1979), 9-10.

[1] A. Ivić, A note on the zero-density estimates for the zeta-function, Archiv Math. 33(1979), 155-164.

[2] A. Ivić, Exponent pairs and the zeta-function of Riemann, Studia Scien. Math. Hungarica, 15(1980), 157-181.

[3] A. Ivić, Exponent pairs and power moments of the zeta-function, Proc. Coll. Soc. Janós Bolyai 34 (Budapest 1981), North-Holland, Amsterdam, in print.

[4] A. Ivić, A zero-density theorem for the Riemann zeta-function, Proc. I.M. Vinogradov Conference (Moscow 1981), Steklov Math. Institute, in print.

[5] A. Ivić, Large values of the error term in the divisor problem, Inventiones Mathematicae, 71 (1983), 513-520.

[1] H. Iwaniec, On mean values for Dirichlet's polynomials and the Riemann zeta-function, J. London Math. Soc. 22(1980), 39-45.

[2] H. Iwaniec, Fourier coefficients of cusp forms and the Riemann zeta-function, Séminaire de Théorie des Nombres, Université Bordeaux 1979/80, exposé no. 18, 36 pp.

[1] H. Iwaniec and M. Jutila, Primes in short intervals, Arkiv för Matematik 17 (1979), 167-176

[1] M. Jutila, On a density theorem of H.L. Montgomery for L-functions, Ann. Scien. Fenn. Ser. A I, 520(1972), 13 pp.

[2] M. Jutila, Zero-density estimates for L-functions, Acta Arith. 32(1977), 52-62.

[3] M. Jutila, Zeros of the zeta-function near the critical line, in "Studies in Pure Mathematics, To the memory of Paul Turán", Birkhäuser, Basel, 1982, in print.

[4] M. Jutila, Riemann's zeta-function and the divisor problem, Institut Mittag-Leffler, report no 13, 1981, 33 pp, in Arkiv för Matematik 21 (1983) 75-96.

[5] M. Jutila, On a formula of Atkinson, Proc. Coll. Soc. Janós Bolyai 34 (Budapest 1981), North-Holland, Amsterdam, in print.

[6] M. Jutila, Transformation formulas for Dirichlet polynomials, to appear in Journal of Number Theory.

[1] A.A. Karacuba, Estimates of trigonometric sums by Vinogradov's method, and some applications (Russian), Proc. Steklov Inst. Math. 119(1971), 241-255.

[2] A.A. Karacuba, Uniform estimates of the error term in Dirichlet's divisor problem (Russian), Izvestija Akad. Nauk SSSR 36(1972), 475-483.

[3] A.A. Karacuba, Elements of analytic number theory (Russian), Moscow, 1975.

[4] A.A. Karacuba, On the distance between consecutive zeros of the Riemann zeta-function on the critical line (Russian), Trudy Mat. Inst. Steklova AN SSSR 157(1981), 49-63.

[1] G. Kolesnik, The distribution of prime numbers in sequences of the form $[n^c]$ (Russian), Mat. Zametki 2(1967), 117-128.

[2] G. Kolesnik, The improvement of the error term in the divisor problem (Russian), Mat. Zametki 6(1969), 545-554.

[3] G. Kolesnik, On the estimation of certain trigonometric sums (Russian), Acta Arith. 25(1973), 7-30.

[4] G. Kolesnik, On the order of Dirichlet L-functions, Pacific J. Math. 82(1979), 479-484.

[5] G. Kolesnik, On the estimation of multiple exponential sums, Recent progress in analytic number theory, Symp. Durham 1979 (vol. 1), 1981, 231-246.

[6] G. Kolesnik, On the order of $\zeta(1/2 + it)$ and $\Delta(R)$, Pacific J. Math. 98(1982), 107-122.

[7] G. Kolesnik, On the number of abelian groups of a given order, J. reine angew. Math. 329(1981), 164-175.

[1] A.F. Lavrik, On the principal term in the divisor problem and the power series of the Riemann zeta-function in a neighborhood of its pole (Russian), Trudy Mat. Inst. Steklova 142(1976), 165-173 and in Proc. Steklov Inst. Math. 1979, issue 3, 175-183.

[1] A.F. Lavrik, M.I. Israilov and Ž. Edgorov, On integrals containing the error term in the divisor problem (Russian), Acta Arith. 37(1980), 381-389.

[1] N. Levinson, More than one third of the zeros of Riemann's zeta-function are on $\sigma = 1/2$, Advances in Math. 13(1974), 383-436.

[1] J.E. Littlewood, Researches in the theory of the Riemann zeta-function, Proc. London Math. Soc. 20(1922), Records XXII-XXVIII.

[1] J. van de Lune, A note on a formula of van der Pol, Acta Arith. 35(1979), 361-366.

[1] S.H. Min, On the order of $\zeta(1/2+it)$, Trans. Amer. Math. Soc. 65(1949), 448-472.

[1] H.L. Montgomery, Zeros of L-functions, Inventiones Math. 8(1969), 346-354.
[2] H.L. Montgomery, Topics in multiplicative number theory, LNM 227, Berlin-Heidelberg-New York, 1971.
[3] H.L. Montgomery, The pair correlation of zeros of the zeta-function, Proc. Symp. Pure Math. 24 AMS, Providence (1973), 181-193.
[4] H.L. Montgomery, Extreme values of the Riemann zeta-function, Comment. Math. Helv. 52(1977), 511-518.
[5] H.L. Montgomery, The analytic principle of the large sieve, Bull. Amer. Math. Soc. 84(1978), 547-567.
[6] H.L. Montgomery, Maximal variants of the large sieve, J. Faculty Science Tokyo 28(1982), 805-812.

[1] H.L. Montgomery and R.C. Vaughan, Hilbert's inequality, J. London Math. Soc. (2) 8(1974), 73-82.

[1] J. Moser, On a certain sum in the theory of the Riemann zeta-function (Russian), Acta Arith. 31(1976), 34-43.
[2] J. Moser, On a theorem of Hardy-Littlewood in the theory of the Riemann zeta-function (Russian), Acta Arith. 31(1976), 45-51 and ibid. 35(1979), 403-404.
[3] J. Moser, The proof of the Titchmarsh hypothesis in the theory of the Riemann zeta-function (Russian), Acta Arith. 36(1980), 147-156.
[4] J. Moser, New consequences of the Riemann-Siegel formula (Russian), Acta Arith. 42, in print.

[1] Y. Motohashi, On Vinogradov's zero-free region for the Riemann zeta-function, Proc. Japan Acad. Ser. A Math. Sci. 54(1978), 300-302.

[2] Y. Motohashi, An elementary proof of Vinogradov's zero-free region for the Riemann zeta-function, Recent progress in analytic number theory, Symp. Durham 1979 (vol. 1), Academic Press 1981, 257-267.

[1] E. Phillips, The zeta-function of Riemann; further development of van der Corput's method, Quart. J. Math (Oxford) 4(1933), 209-225.

[1] K. Prachar, Primazahlverteilung, Berlin-Göttingen-Heidelberg, 1957.

[1] K. Ramachandra, A simple proof of the mean fourth power estimate for $\zeta(1/2+it)$ and $L(1/2+it,\chi)$, Ann. Scuola Norm. Sup. Pisa 1(1974), 81-97.

[2] K. Ramachandra, Some new density estimates for the zeros of the Riemann zeta-function, Ann. Acad. Sci. Fenn. Ser. A I Math. 1(1975), 177-182.

[3] K. Ramachandra, Applications of a theorem of Montgomery and Vaughan to the zeta-function, J. London Math. Soc. (2)10(1975), 482-486.

[4] K. Ramachandra, Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series I, Hardy-Ramanujan J. 1(1978), 1-15; II ibid. 3(1980), 1-24.

[5] K. Ramachandra, Some remarks on a theorem of Montgomery and Vaughan, J. Number Theory 11(1980), 465-471.

[6] K. Ramachandra, Progress towards a conjecture on the mean value of Titchmarsh series, Recent progress in analytic number theory, symp. Durham 1979 (vol. 1) Academic Press 1981 (303-318).

[1] R.A. Rankin, Van der Corput's method and the theory of exponent pairs, Quart. J. Math. (Oxford) (2) 6(1955), 147-153.

[1] A. Reich, Dirichletreihen und gleichverteilte Folgen, Analysis, to appear.

[2] A. Reich, Oberwolfach Vortragsbuch, Tagungsbericht 41(1982), 29-30.

[1] H.-E. Richert, Verschärfung der Abschätzung beim Dirichketschen Teilerproblem, Math. Zeit. 58(1953), 204-218.

[2] H.-E. Richert, Über Dirichletreihen mit Funktionalgleichungen, Publications Inst. Math. Belgrade 11(1957), 73-124.

[3] H.-E. Richert, Einführung in die Theorie der Starken Rieszschen Summierbarkeit von Dirichletreihen, Nachr. Akad. Wiss. Göttingen (Math.-Physik) 1960, 17-75.

[4] H.-E. Richert, Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen $\sigma = 1$, Math. Annalen 169(1967), 97-101.

[1] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebener Grösse, Monatsberichte Akad. Berlin 1859, 671-680.

[1] J.B. Rosser, J.M. Yohe and L. Schoenfeld, Rigorous computation and the zeros of the Riemann zeta-function, Cong. Proc. Int. Federatiom Information Process 1968, Washington-New York 1969, 70-76.

[1] A. Selberg, Contributions to the theory of the Riemann zeta-functioh, Archiv for Math. og Naturv. B 48(1946), 89-115.

[1] Shi-Tuo Lou, A lower bound for the number of zeros of Riemann's zeta-function on $\sigma = 1/2$, Recent progress in analytic number theory, Symp. Durham 1979 (vol. 1), 1981, 319-325.

[1] B.R. Srinivasan, Lattice point problems of many-dimensional hyperboloids II, Acta Arith. 8(1963), 173-204.

[2] B.R. Srinivasan, Lattice point problems of many-dimensional hyperboloids III, Math. Annalen 160(1965), 280-311.

[3] B.R. Srinivasan, On the number of abelian groups of a given order, Acta Arith. 23(1973), 195-205.

[1] E.C. Titchmarsh, On van der Corput's method and the zeta-function of Riemann I, Quart. J. Math. (Oxford) 2(1931), 161-173; II ibid. 2(1931), 313-320.

[2] E.C. Titchmarsh, On Epstein's zeta-function, Proc. London Math. Soc. 36(1934), 485-500.

[3] E.C. Titchmarsh, The lattice points in a circle, Proc. London Math. Soc. 38 (1934), 96-115.

[4] E.C. Titchmarsh, On van der Corput's method and the zeta-function of Riemann IV, Quart. J. Math. (Oxford) 5(1934), 98-105; V ibid. 5(1934), 195-210.

[5] E.C. Titchmarsh, The approximate functional equation for $\zeta^2(s)$, Quart. J. Math. (Oxford) 9(1938), 109-114.

[6] E.C. Titchmarsh, On the order of $\zeta(1/2+it)$, Quart. J. Math. (Oxford) 13(1942), 11-17.

[7] E.C. Titchmarsh, Introduction to the theory of Fourier integrals, Oxford, 1948.

[8] E.C. Titchmarsh, The theory of the Riemann zeta-function, Oxford, 1951.

[1] K.-C. Tong, On divisor problems, J. Chinese Math. Soc. 2(1953), 258-266.

[2] K.-C. Tong, On divisor problems III, Acta Math. Sinica 6(1956), 515-541.

[1] P. Turán, On some recent results in the analytical theory of numbers, Proc. Symp. Pure Math. XX 1969, Institute of Number Theory 1971, 359-374.

[1] G.F. Voronoï, Sur une fonction transcendante et ses applications à la sommation de quelques séries, Ann. École Normale (3)21(1904), 207-268; ibid. 459-534.

[2] G.F. Voronoï, Sur le développement, à l'étude des fonctions cylindriques, des sommes doubles $\sum f(pm^2+2qmn+rn^2)$, où $pm^2+2qmn+rn^2$ est une forme positive à coefficients entiers, (Verh. III Math. Kongress Heidelberg), Teubner, Leipzig 1905, 241-245.

[1] A. Walfisz, Zur Abschätzung von $\zeta(1/2+it)$, Nachr. Ges. Wiss. Göttingen. math. phys. Klasse 1924, 155-158.

[2] A. Walfisz, Über zwei Gitterpunkprobleme, Math. Annalen 95(1925), 69-83.

[3] A. Walfisz, Weylsche Exponentialsummen in der neuren Zahlentheorie, Berlin, 1963.

[1] G.N. Watson, Theory of Bessel Functions, Cambridge, 1944.

[1] R. Wiebelitz, Über approximative Funktionalgleichungen der Potenzen der Riemannschen Zeta-funktion, Math. Nachr. 6(1951-52), 263-270.

[1] Yüh Ming-i, A divisor problem, Acta Math. Sinica 8(1958), 496-506.

[2] Yüh Ming-i and Wu Fang, On the divisor problem for $d_3(n)$, Scien. Sinica 11(1962), 1055-1060.

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