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METRIC SUBSPACES OF L^1

by

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Summary : It is well known that a normed space E is a normed subspace of L^1 if and only if the metric on E is of negative type; moreover the finite dimensional normed subspaces of L^1 have been extensively studied (zonoïds, zonotopes,...) .

It is not so simple to recognize the metric subspaces of L^1 by means of inequalities (negative type does not suffice) . Moreover a metric subspace of L^1 (and L^1 itself) is always a metric subspace of a σ-algebra endowed with the symmetric difference metric (for some measure) ; thus the study of the metric subspaces of L^1 will have a marked combinatorial character (related to codes, designs,...) .

We will study here (*) with full proofs and some details the convex cone of all metrics d on a given set X such that the space (X,d) embeds into L^1 (paragraph 1) and the dual cone of all inequalities (with a given number of arguments) which are valid for all metric subspaces of L^1 (paragraph 2).

(*) The two paragraphs of this paper were written a year ago as chapters of a book on metric spaces projected (and not achieved) by the two authors; they are given here with only slight changes.

Sommaire : Les sous espaces métriques de L^1 ont jusqu'à présent moins retenu l'attention que les sous espaces normés de L^1; de fait il n'est pas aussi simple de les reconnaitre au moyen d'inégalités (ainsi, que la distance soit de type négatif suffit pour les espaces normés, mais ne suffit plus pour les espaces métriques) .

Par ailleurs l'étude des sous espaces métriques de L^1 a un caractère combinatoire assez prononcé (relié aux codes, aux plans d'expérience,...) à cause du simple fait suivant : un sous espace métrique de L^1 (et L^1 lui même) est toujours un sous espace métrique d'une σ-algèbre munie de la distance de la différence symétrique (mesurée par une certaine mesure positive) .

Le présent article (*) étudie, avec des démonstrations détaillées, le cone convexe de tous les écarts d sur un ensemble donné X tels que l'espace (X,d) soit un sous espace métrique de L^1 (paragraphe 1) et le cone dual de toutes les inégalités (à un nombre fixé d'arguments) qui sont vraies dans tout sous espace métrique de L^1 (paragraphe 2) .

(*) Les deux paragraphes formant cet article ont été écrits il y a un an comme chapitres d'un livre sur les espaces métriques, livre projeté (et maintenant abandonné) par les deux auteurs; l'usage de la langue anglaise provient aussi de ce livre .

INTRODUCTION : The present paper is a study of the basic properties of the metric subspaces of L^1 (§1) and of the inequalities they satisfy (§2) . Each paragraph will have its own introduction and its own bibliography (let us recall that they were written as chapters of a projected and not achieved book) .

Let us give here nevertheless a list of the more important notions to be introduced in §1 and §2 (with their location):
in contrast with the metric subspaces of L^2, the (possible) representation of a metric space as a metric subspace of L^1 is generally not unique ; actually it is always possible to represent a metric subspace of L^1 as a family of subsets (of a set T) endowed with the symmetric difference metric (see below p 5) relative to a nonnegative measure μ . Such a representation will be called a realization (p 7) and the total mass of the measure μ is the size (p 7) of the realization ; if moreover the measure μ takes its values in $\frac{1}{\eta}\mathbb{N}$, then the realization is called a realization at scale η (p 22) . A realization at scale 1 is nothing else than an isometric embedding into a hypercube (p 4); it can be considered as an incidence structure (p 8) .

A semimetric d (see note p 2) on a set X is said to be L^1-embeddable (p 6) if the space (X,d) is a subspace (in the extended sense of p 5) of some space L^1 . The set of all L^1-embeddable semimetrics on the set X will be denoted P(X) (p 8); it is a convex cone ; its extremal rays are defined by the dichotomies (p 10) and it enjoys some finiteness property (p 18) .

For $|X| = m$, the dual cone P'(X) is the cone of all inequalities valid in L^1 of order m (pp 19 et 28); a metric space is a metric subspace of L^1 if and only if it satisfies all inequalities valid in L^1 . The more important examples of these inequalities are the polygonal (or hypermetric) inequalities (p 29), which include the triangular inequality (p 31), the pentagonal inequality (p 32) and the negative type inequalities (p 30); other examples could be find p 46 (the polygonal inequalities were introduced by Deza 1960 and independently rediscovered by Kelly 1967).

Let us recall to end that a metric subspace of a hypercube satisfies, beside these inequalities, the condition of perimeter (a congruence condition, p 23)

§1 THE CONVEX CONE OF ALL L^1-METRICS

The purpose of this paragraph is to introduce the following concepts : subspaces of L^1 , hypercubes, realization, size, scale, and to give the most general properties involving these notions. It will contain three sections :

1.A will contain the notations and definitions about metric subspaces of L^1 (and thus is important in order to read the other parts of this survey); it contains also the first properties of these spaces;
no definite reference is used for measure theory (we suggest, in french, the chapters 1,2,3 of the book of Meyer [21] and, in english, Royden [24] or Cohn [13]); in fact a reader interested only in finite combinatorics may assume that each measure is discrete ;

1.B is a study of the convex cone of all semimetrics d (on a given, possibly infinite, set X) such that (X,d) is a subspace of L^1 ;
the results will become far more easier if the set X is assumed to be finite; the results in this section which can be useful in finite combinatorics are 1.26, 1.27, 1.28, 1.30, 1.33, 1.40 and 1.42, excluding the results of finiteness; no definite reference is used for convex analysis (one can see, in french, Bourbaki [10] , in english, the three volumes of Choquet [12]) ;

1.C contains further informations (antipodal extension, perimeter condition,...) and include an important (even if easy) Lemma on the scale of a finite, integer valued metric space.

The whole paragraph is due to the collaboration of the two authors, except 1.B which is due to P.Assouad alone .

1.A Metric subspaces of L^1 and realizations :

Our problem will be to study the possibility of isometric embedding of metric spaces into same basic metric spaces, which will be generally some subspaces of a L^p-space. We have to recall some definitions; since they are not really surprising, the best is to go now to the first results (1.11) and to come back to the definitions only when needed. Let us define $L^p(T,\mathcal{B},\nu)$ (or $L^p(\nu)$ for short) :

(1.1) Let $(T,\mathcal{B})$ be a measurable space (i.e. the pair of a set T and of a σ-algebra $\mathcal{B}$ of subsets of T) and ν a nonnegative measure on $(T,\mathcal{B})$ (see 1.2 below); such a triple $(T,\mathcal{B},\nu)$ (with $\nu \geqslant 0$) will be called a measure space . We fix $p \in [1,+\infty[$. Then $L^p(T,\mathcal{B},\nu)$ is the vector space of all measurable applications $f : (T,\mathcal{B}) \longrightarrow \mathbb{C}$ such that $\int_T |f(t)|^p \nu(dt) < +\infty$

It is endowed with the seminorm :

$$f \in L^p(T,\mathcal{B},\nu) \longrightarrow \|f\|_{L^p(\nu)} = \left(\int_T |f(t)|^p \, \nu(dt)\right)^{1/p}$$

and thus with the following semimetric (*) :

$$f,g \longrightarrow \|f-g\|_{L^p(\nu)}$$

In fact it is more classical to denote by $L^p(T,\mathcal{B},\nu)$ only the corresponding normed space (or metric space) , which is obtained by identifying two measurable functions $f_1,f_2 : (T,\mathcal{B}) \longrightarrow \mathbb{C}$ when $f_1 = f_2$ almost everywhere.

(1.2) Let us precise that a nonnegative measure ν on $(T,\mathcal{B})$ is for us an application $\nu : \mathcal{B} \longrightarrow [0,+\infty]$ satisfying $\nu(\emptyset) = 0$ and $\nu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \nu(A_i)$ for each sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint elements of $\mathcal{B}$; in other words it is a nonnegative σ-additive measure, but not necessarily finite or σ-finite.

(1.3) For example, we will use frequently the following discrete measure space (the reader only interested in finite combinatorics may assume that no other measure is used) :

let I be a set of cardinality n (finite or infinite) and

let $i \longrightarrow \mu_i$ be an application of I into $[0,+\infty]$;

let us take $\mathcal{A} = \{0,1\}^I$ (or 2^I) the σ-algebra of all subsets of I ; then we define a nonnegative measure on $(I,\mathcal{Q})$ by setting $\mu(A) = \sum_{i\in A} \mu_i$ for each $A \in \mathcal{Q}$; in this case μ will be called a discrete measure on I ; for each $i \in I$, the number μ_i will be called the weight of the measure μ at the point i . Two particular cases are important :

(1.3.1) fix $j \in I$ and assume that one has $\mu_j = 1$ and $\mu_i = 0$ for all $i \in I \setminus \{j\}$; then the measure μ defined as above is called the Dirac mass at the point j and will be denoted δ_j ; we will use also the notation $1_{j\in A}$ for $\delta_j(A)$;

(1.3.2) assume that one has $\mu_i = 1$ for all $i \in I$; then the measure μ defined above is called the cardinality measure ; in other words it is the measure μ defined by $\mu(A) = |A|$ for all $A \in \mathcal{Q}$ (i.e. $\mu(A)$ is the cardinality of A if A is finite, $\mu(A) = +\infty$ otherwhise); the measure space $(I,2^I,|\ |)$ will be denoted for short I .

The ℓ^p-spaces are a particular case of L^p-spaces :

(1.4) (the spaces ℓ^p_I , $\ell^p_{(n)}$ and ℓ^p) :

Let I be a set of cardinality n (finite or infinite) ; then $L^p(I)$ is usually denoted $\ell^p(I)$ (or $\ell^p(n)$ if we insist only on the cardinality of I) ; if n is the cardinality of the set of all integers, the space $\ell^p(n)$ will be simply denoted ℓ^p .

Indeed , we are more interested in the spaces $L^1(\mu)$ (i.e. in the case p=1) . The next examples of our list will be clearely subspaces of some space $L^p(\mu)$; we begin by two discrete examples :

(1.5) (the regular grill P^n_k of dimension n and with k) : let us take $k \in \mathbb{N} \cup \{+\infty\}$ and I a set of cardinality n (finite or infinite); the regular grill P^n is the set $x \in \{0,\dots,k\}^I \;\Big|\; \sum_{i\in I} |x_i| < +\infty$ (or the set $x \in \mathbb{Z}^I \;\Big|\; \sum_{i\in I} |x_i| < +\infty$ in the case $k = +\infty$) endowed with the following metric :

$d(x,y) = \sum_{i\in I} |x_i - y_i|$ (a metric with infinite values on $\mathbb{Z}^I$).

Therefore it has also the following equivalent definitions (for n finite):

- it is the set of vertices of the graph P_k^n with the pathmetric (or geodesic metric) ; the graph P_k^n is the cartesian product of n pathes of length k (see also below 1.18) ;

- it is also the metric subspace of all elements of $\ell^1_{(n)}$ having coordinates in $\{0,\dots,k\}$.

The next example is more important :

Definition 1.6 (the *hypercube of dimension n at scale* η , denoted $\frac{1}{\eta} K_2^n$) :

let us take $\eta \in]0,+\infty[$ and a set Ω of cardinality n (finite or infinite); the hypercube of dimension n at scale η is the set $2^{(\Omega)}$ of all finite subsets of Ω endowed with the metric defined as follows :

$$d(A,B) = \frac{1}{\eta} |A \Delta B| \quad \text{for all finite } A,B \subset \Omega$$

If we take $\eta = 1$, then it will be called shortly the *hypercube of dimension n* and denoted K_2^n .

Therefore K_2^n has also the following equivalent definitions (for n finite).

- it is the set of all binary words of length n with the Hamming's metric (as used in coding theory) ;

- it is the set of vertices of a unit hypercube of R^n , endowed with the path metric ;

- the structure of graph mentionned above is also the cartesian product of n graphs K_2 and thus it will justify the notation ;

- it is also exactly the regular grill P_1^n (but the particular importance of K_2^n among regular grills will be shown in Proposition 1.11).

(1.7) One shows that $\frac{1}{\eta}$ K_2^n) is a metric subspace of a space L^1 by using the following observation on symmetric difference :

$$1_{A\Delta B} = |1_A - 1_B| \quad \text{for all} \quad A,B \subset \Omega .$$

Another important metric subspace of L^1 (in fact a semimetric space) is the following continuous analog of K_2^n :

Definition 1.8 (the space $K(\Omega,\mathcal{A},\mu)$ or shortly $K(\mathcal{A},\mu)$, taken from [4])

Let $(\Omega,\mathcal{A},\mu)$ be a measure space ($\mu \geq 0$, see 1.2) ; then $K(\mathcal{A},\mu)$ is the set $\{A \in \mathcal{A} \mid \mu(A) < +\infty\}$ endowed with the following semimetric :

$$d(A,B) = \mu(A \,\Delta\, B) \quad \text{for all} \quad A,B \in \mathcal{A} \text{ having finite measure}$$

(Naturally $\frac{1}{n} K_2^n$ is only a special case of $K(\mathcal{A},\mu)$).

Our notations concerning embeddings will be taken from [4] :

Definition 1.9 Let X, Y be two sets and let δ be an application from $Y \times Y$ into $\mathbb{R}$ (we will call such a pair (Y,δ) a space , supported by Y). Let f be an application (not necessarily injective) of X into Y ; then we define the inverse image of δ by f (denoted $\delta \circ f$) by setting : $\delta \circ f\,(x,x') = \delta(f(x),f(x'))$ for all $x,x' \in X$.

Let d be an application from $X \times X$ into $\mathbb{R}$; the space (X,d) is called embeddable into the space (Y,δ) if there is an application $f : X \longrightarrow Y$ such that $d = \delta \circ f$; in this case, we will say that f is an (isometric) embedding of (X,d) into (Y,δ) ; we will say equivalently (even if f is not injective) that the space (X,d) is a subspace of (Y,δ).

If X is a subset of Y and f is the natural injection of X into Y , then $\delta \circ f$ is the restriction of δ to X and is denoted $\delta_{|X}$.

In fact we are mainly interested in the subspaces of the spaces $L^1(\mu)$, P_k^n , $\frac{1}{n} K_2^n$ and $K(\mathcal{A},\mu)$ described above ; thus we will specify the notations about embeddings into these spaces :

(1.10) Let d be a semimetric on a set X ; let us take $p \in [1,+\infty[$ and $\eta \in]0,+\infty[$:

(1.10.1) the space (X,d) (or the semimetric d) is said to be L^p-embeddable if there is a measure space $(T,\mathfrak{B},\nu)$ (with $\nu \geqslant 0$) such that (X,d) is a subspace (in the sense of 1.9) of the space $L^p(T,\mathfrak{B},\nu)$;

(1.10.2) the space (X,d) is said to be h-embeddable at scale η is there is an integer or (in the infinite case) a cardinal n such that (X,d) is a subspace (in the sense of 1.9) of the space $\frac{1}{\eta}\ K_2^n$;

for $\eta = 1$, we will say that (X,d) is h-embeddable (h is the first letter of "hypercube" and of "Hamming").

We turn now to consider only subspaces of L^1 ; we have observed that $K(\Omega,\mathfrak{A},\mu)$ is a subspace of a space L^1 (precisely of $L^1(\Omega,\mathfrak{A},\mu)$) and that K_2^n is a subspace of a regular grill (precisely it is P_1^n); we will now see that the converse holds :

Proposition 1.11 : Every space $L^1(T,\mathfrak{B},\nu)$ is a subspace of some space $K(\mathfrak{A},\mu)$. Every regular grill P_k^n is a subspace of some hypercube.

proof : (1.11.1) Let $(T,\mathfrak{B},\nu)$ be a measure space and consider $\Omega = T \times \mathbb{R}$, $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{R}$, $\mu = \nu \otimes \lambda$ where $\mathfrak{R}$ is the σ-algebra of all Borel subsets of $\mathbb{R}$ and λ is the Lebesgue measure on $\mathbb{R}$; for each $f \in L^1(T,\mathfrak{B},\nu)$, we set

$$E(f) = \{(t,v) \in T \times \mathbb{R} \mid v > f(t)\} \quad \text{(the epigraph of } f) ;$$

now we see that the application $f \longrightarrow E(f) \,\Delta\, E(0)$ is an (isometric) embedding of $L^1(T,\mathfrak{B},\nu)$ into $K(\mathfrak{A},\mu)$.

(1.12.1) We take for example k finite ; the proof is the same, except tha we take for each $x \in \{0,\dots,k\}^I$ (where I is a set of cardinality n):

$$E(x) = \{(i,n) \in I \times \{1,\dots,k\} \mid n > x_i\} \quad ;$$

thus we see that (for k finite) the regular grill P^n_k embeds into the hypercube K^m_2 with $m = n(k-1)$;
obviously the same is true for P^n_∞ . □
The above observation (1.11.1) can be find for example in Oxtoby [23] p.44 and is surely much older (Nikodym ?) ; the idea to use it in studying metric subspaces of L^1 comes from [4].

We will see that the good way to embed an L^1-embeddable metric space in L^1 is to embed it in space $K(\mathcal{A},\mu)$;
thus it is important to have a precise terminology about embeddings int $K(\mathcal{A},\mu)$:

Definition 1.12 : Let (X,d) be an L^1-embeddable semimetric space ; then an embedding f of (X,d) into $K(\Omega,\mathcal{A},\mu)$ will be called a realization of (X,d) into $(\Omega,\mathcal{A},\mu)$ $(\mu \geq 0)$. The size of this realization f is the number $\sigma(f) = \mu(\Omega)$ (in other words the total mass of the measure μ).

(1.13) More explicitely a realization f of (X,d) into a measure space $(\Omega,\mathcal{A},\mu)$ $(\mu \geq 0)$ is the following :

an application $f : x \longrightarrow A_x$ from X into $\{A \in \mathcal{A} \mid \mu(A) < +\infty\}$ which satisfies : $d(x,x') = \mu(A_x \Delta A_{x'})$ for all $x,x' \in X$; in this setting it is useful to consider the set $A = \{(x,\omega) \in X \times \Omega \mid \omega \in A_x\}$ as an incidence structure on $X \times \Omega$ (see Dembowski [14]) :
for each $\omega \in \Omega$ the set $A^\omega = \{x \in X \mid (x,\omega) \in A\}$ is called a block of the realization f .

Embedding into $\frac{1}{\eta} K^n_2$ is only a special (but important) case :

Definition 1.14 : Let (X,d) be a semimetric space which is h-embeddable at scale η ; then an embedding f of (X,d) into $\frac{1}{\eta} K^n_2$ will be called a h-realization of (X,d) at scale η , shortly a h-realization if $\eta=1$;

moreover it will be called a h-realization into Ω , in order to precise that the space $\frac{1}{\eta} K_2^n$ is supported by the set 2^Ω . The size of this h-realization f is (according to 1.12) the number $\sigma(f) = \frac{1}{\eta} n$ if n is finite, $\sigma(f) = +\infty$ otherwise.

(1.15) More explicitely a h-realization f of (X,d) at scale η into Ω is the following :

an application $f : x \to A_x$ from X into the set of all finite subsets of a given set Ω (of cardinality n) which satisfies :

$\eta\, d(x,x') = |A_x \,\Delta\, A_{x'}|$ for all $x,x' \in X$;

according to 1.13 , we define the corresponding incidence structure

$A = \{(x,\omega) \in X \times \Omega \mid \omega \in A_x\}$ and the corresponding blocks

$A^\omega = \{x \in X \mid (x,\omega) \in A\}$ for each $\omega \in \Omega$;

we note that the elements of X are sometimes called treatements (resp. points) and the elements of Ω itself blocks (resp. lines); this terminology comes from designs of experiment in statistics (resp. incidence structures in geometry) ; see Dembowski [14].

(1.16) In some cases it will be more convenient to describe a h-realization f into Ω (when X and Ω are finite) by its incidence structure given in the following way :

the incidence structure is written as a $X \times \Omega$-matrix a (i.e. the lines are indexed by X and the rows are indexed by Ω) with only binary entries (the entry $a_{x,\omega}$ is 1 if $(x,\omega) \in A$ and is 0 otherwise) ;

in this case we will say that the h-realization f is given in incidence form.

We need now some notations for the set of all embeddable semimetrics :

Definition 1.17 The set of all L^1-embeddable (resp. of all h-embeddable) semimetrics on a given set X will be denoted P(X) (resp. h-P(X)).

Let us recall the concept of direct sum :

Definition 1.18 : Let (X_1,d_1) and (X_2,d_2) be two spaces (in the sense of 1.9).

The *direct sum* of (X_1,d_1) and (X_2,d_2) is the pair (X,d) denoted $(X_1,d_1) \oplus (X_2,d_2)$ and defined as follows :

$X = X_1 \times X_2$ and d is the following application from X^2 into $\mathbb{R}$:

$\forall\, x_1,x_1' \in X_1$, $\forall\, x_2,x_2' \in X_2$,

$$d((x_1,x_2),(x_1',x_2')) = d_1(x_1,x_1') + d_2(x_2,x_2') \quad ;$$

d is denoted $d_1 \oplus d_2$ and called the *direct sum* of d_1 and d_2.

Proposition 1.19 : Let d_1 be a semimetric on a set X_1 and d_2 a semimetric on a set X_2 . Let us assume that d_1 and d_2 are L^1-embeddable (resp. h-embeddable). Then $d_1 + d_2$ is L^1-embeddable (resp. h-embeddable).

proof : It is clear that we have :

$$L^1(T_1,\mathfrak{B}_1,\nu_1) \oplus L^1(T_2,\mathfrak{B}_2,\nu_2) = L^1(T,\mathfrak{B},\nu)$$

where T is the disjoint union of T_1 and T_2 , $\mathfrak{B}$ the σ-algebra whose restriction to T_1 (resp. T_2) is $\mathfrak{B}_1$ (resp. $_2$) and ν the measure whose restriction to T_1 (resp. T_2) is ν_1 (resp. ν_2) ;

we have also $P^n \oplus P^m = P^{n+m}$ for all $\in \mathbb{N} \cup \{0\}$. □

Corollary 1.20: Let X be a set.

$P(X)$ is a *convex cone* (i.e. $d_1,d_2 \in P(X)$, $\alpha \in [\,0,\infty\,[$. implies $d_1 + d_2$, $\alpha\, d_1 \in P(X)$).

h-$P(X)$ is an *additive semi group* (i.e. $d_1,d_2 \in$ h-$P(X)$ implies $d_1 + d_2 \in$ h-$P(X)$)

proof : Let f be the diagonal application of X into $X \times X$ i.e. :

$$\forall x \in X \quad , \quad f(x) = (x,x).$$

Then f is an embedding of (X,d_1+d_2) into $(X,d_1) \oplus (X,d_2)$. Let us assume that d_1 and d_2 are L^1-embeddable (resp. h-embeddable). Then (using Proposition 1.19) $d_1 + d_2$ is L^1-embeddable (resp. h-embeddable) since a subspace of a subspace is a subspace.
Moreover it is clear that, if d is L^1-embeddable and $\alpha \in [0,\infty[$, then αd is L^1-embeddable. □

We will define now some particular semimetrics on X which will be shown in 1.25 to define certain extremal rays (in fact all) of the cone P(X) :

Definition 1.21 : A dichotomy on a set X is a semimetric π on X having the following form :

there is a partition X_1, X_2 of X (i.e. $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$) such that:

$\pi(x,x') = 1$ for $x \in X_1$, $x' \in X_2$ and $x \in X_2$, $x' \in X_1$,

$\pi(x,x') = 0$ otherwise.

The set of all dichotomies on a set X is denoted $\Pi(X)$.

We give here three criterion (taken from [4]) for a semimetric to be a dichotomy ; the proof (easy) is left to the reader :

Lemma 1.22 : Let X be a set and d be an application from X^2 into $\mathbb{R}$; then the following are equivalent :

(1.22.1) d is a dichotomy on X ;

(1.22.2) the space (X,d) is a subspace (in the sense of 1.9) of a hypercube K_2 of dimension 1 ;

(1.22.3) d has values in $\{0,1\}$ and satisfies the condition of perimeter (i.e., by definition, $d(x_1,x_2) + d(x_2,x_3) + d(x_3,x_1)$ is even for each $x_1,x_2,x_3 \in X$) ;

(1.22.4) d has the following form :

$$d(x,x') = \frac{1}{2}\ [1 - \varepsilon(x)\varepsilon(x')] \quad \text{for all} \quad x,x' \in X ,$$

where ε is an application X into $\{-1,1\}$.

(1.23) Let us observe that 1.22.2 implies that each dichotomy is h-embeddable (and thus L^1-embeddable). When X is finite, then there are exactly $(2^{|X|-1} - 1)$ non zero dichotomies on X and the convex cone generated by these dichotomies is often called the Hamming cone on X (it is the cone studied by [15], [20], [7] and it is a polyhedral cone).

Let us denote $D(X)$ the convex cone of all semimetrics on a given set X .

It was shown in Deza [17] that each non zero dichotomy, on a finite set X , belongs to an extremal ray of the Hamming cone on X .

A more precise observation is made in Avis [8]:

Lemma 1.24 : Let X be a set. Then each dichotomy π on X belongs to an extremal ray of the convex cone $D(X)$ of all semimetrics on X .

proof : Let π be a dichotomy on X and let d,d' be elements of $D(X)$ such that $d + d' = \pi$. Assume that the dichotomy π is defined by the partition (X_1,X_2) of the set X (see 1.21).

Then for each $x,x' \in Y_1$ (resp. Y_2) we have $\pi(x,x') = 0$ and thus $d(x,x') = 0$. Hence the semimetric d identifies the points of Y_1 (resp. Y_2).

Therefore there is a nonnegative real α such that $d = \alpha\,\pi$.

Thus π belongs to an extremal ray of $D(X)$. □

(see Avis [8] for other extremal rays of $D(X)$).

(1.25) The dichotomies on X are elements of $P(X)$ which belongs to some extremal rays of the convex cone $D(X)$ (which include $P(X)$) ; thus they belong to an extremal ray of $P(X)$. We will see in §B that each extremal ray of $P(X)$ contains a non zero dichotomy ; as a consequence, if X is finite, the Hamming cone on X (see 1.23) is nothing else than $P(X)$.

1.B Extremal rays and finiteness :

We will see now that each L^1-embeddable (resp. h-embeddable) semimetric is an integral of dichotomies with respect of a nonnegative measure (resp. a nonnegative integer measure). Let us define first multiplicities :

(1.26) Let (X,d) be a L^1-embeddable semimetric space ;
let us fix $f : x \longrightarrow A_x$ a realization of (X,d) into $(\Omega,\mathcal{A},\mu)$ and take the corresponding blocks (see 1.13) $A^\omega = \{x \in X \mid \omega \in A_x\}$. Consider the set $2^X = \{0,1\}^X$ of all subsets of X ; 2^X will be endowed with the σ-algebra $\mathcal{A}_X$ generated by all projections from 2^X onto $\{0,1\}$ and we define an application j_X from X into $\mathcal{A}_X$ by setting :

$$j_X(x) = \{Y \subset X \mid x \in Y\} \quad \text{for each } x \in X .$$

We note that the application $f^\star : \omega \longrightarrow A^\omega$ is measurable from $(\Omega,\mathcal{A})$ into $(2^X, \mathcal{A}_X)$; the nonnegative measure $\nu_{f,\mu} = f^\star(\mu)$ will be called the multiplicity measure of the realization f (of (X,d) into $(\Omega,\mathcal{A},\mu)$).

Proposition 1.27 : Let (X,d) be a L^1-embeddable semimetric space ;
let us take $f : x \longrightarrow A_x$ a realization of (X,d) into $(\Omega,\mathcal{A},\mu)$.
Then $j_X : x \longrightarrow j_X(x)$ will be a realization of (X,d) into $(2^X,\mathcal{A}_X,\nu_{f,\mu})$ and it will be called the multiplicity realization of (X,d) corresponding to f.

proof : We have for each $x,x' \in X$ $(x \neq x')$:

$$\nu_{f,\mu}(j_X(x) \,\Delta\, j_X(x')) = \nu_{f,\mu}\{Y \subset X \mid |Y \cap \{x,x'\}| = 1\}$$

$$= \mu\{\omega \mid |A^\omega \cap \{x,x'\}| = 1\} = \mu(A_x \,\Delta\, A_{x'}) = d(x,x') . \quad \square$$

(1.28) We give a more explicit description of the multiplicity measure in the case : (X,d) is h-embeddable and $f : x \to A_x$ is a h-realization of (X,d) into Ω . In this case the multiplicity measure will be shortly denoted ν_f and is the discrete measure on 2^X defined as follows : for each $Y \in 2^X$, the weight of the measure ν_f at the point Y is the integer $|\{\omega \mid A^\omega = Y\}|$ i.e. the number of blocks equal to Y (this justifies our terminology).

In fact we will get easily from the multiplicity measure a measure on the set $\Pi(X)$ of all dichotomies having d for resultant :

(1.29) The set $\Pi(X)$ of all dichotomies on the set X will be endowed with the σ-algebra $\mathfrak{B}_X$ generated by all coordinate applications $\pi \to \pi(x,x')$ from $\Pi(X)$ into $\{0,1\}$; for each $Y \in 2^X$ we consider the dichotomy π_Y defined by : $\pi_Y(x,x') = |1_Y(x) - 1_Y(x')|$ for all $x,x' \in X$; the application $g : Y \to \pi_Y$ is measurable from $(2^X, \mathcal{A}_X)$ into $(\Pi(X), \mathfrak{B}_X)$.

Proposition 1.30 [4] : (1.30.1) Let (X,d) be a L^1-embeddable semimetric space. We fix a realization f of (X,d) into $(\Omega, \mathcal{A}, \mu)$ and we set $\nu = g(\nu_{f,\mu})$ (it is a nonnegative measure on $(\Pi(X), \mathfrak{B}_X)$) . Then we have :

$$\forall x,x' \in X \quad , \quad d(x,x') = \int_{\Pi(X)} \pi(x,x')\, \nu(d\pi) \quad ;$$

moreover if f is a h-realization at scale η , then $\eta\nu$ is a discrete measure having values in $\mathbb{N} \cup \{+\infty\}$ (such a measure will be called an integer measure)

(1.30.2) Conversely let ν be a nonnegative measure on $(\Pi(X), \mathfrak{B}_X)$; we set for each $x,x' \in X$, $d(x,x') = \int_{\Pi(X)} \pi(x,x')\, \nu(d\pi)$

(we will write shortly $d = \int \pi \; \nu(d\pi)$) ;

then (X,d) is L^1-embeddable ; if moreover $\eta\nu$ is an integer measure (for some $\eta \in]0,+\infty[$) , then (X,d) is h-embeddable at scale η .

proof : (1.30.1) From 1.27 , we have for each $x,x' \in X$:

$$d(x,x') = \nu_{f,\mu}(j_X(x) \,\Delta\, j_X(x')) = \nu_{f,\mu}\{Y \mid \pi_Y(x,x') = 1\}$$
$$= \int_{\Pi(X)} \pi(x,x')\ \nu(d\pi) \quad ;$$

if moreover d is h-embeddable at scale η , then we see from 1.28 that $\eta\nu$ is an integer measure.

(1.30.2) Now let ν be a nonnegative measure on $(\Pi(X),\mathfrak{B}_X)$ and set $d = \int \pi\ \nu(d\,\pi)$. We fix a point $s \in X$ and we define for each $x \in X$ an element A_x of $\mathfrak{B}_X$ by setting

$$A_x = \{\pi \in \Pi(X) \mid \pi(x,s) = 1\} ;$$

it is clear that $x \longrightarrow A_x$ is a realization of (X,d) into $(\Pi(X),\mathfrak{B}_X,\nu)$.
If moreover $\eta\nu$ is an integer measure, we call n the cardinality of $\Pi(X)$ and we define for each $x \in X$ an element f_x of P^n_∞ by setting :
$f_x(\pi) = \nu(\{\pi\})\ \pi(x,s)$;
it is clear that $x \longrightarrow f_x$ is an (isometric) embedding of $(X,\eta d)$ into P^n_∞ and the result follows from 1.11. □

Let us make an observation about dichotomies :

(1.31) Let π be a dichotomy on X and take $x_1,x_2,x_3 \in X$;
then one has $\pi(x_1,x_2) = \pi(x_1,x_2)\ [\pi(x_1,x_3) + \pi(x_2,x_3)]$
and $\pi(x_1,x_2)\ \pi(x_2,x_3)\ \pi(x_3,x_1) = 0$;
these obvious relations can be interpreted as the Pasch's axiom for dichotomies

The following result describes all extremal rays of the cone $P(X)$ (this result is due to [4] for infinite X , and to [8],[17] for finite X) :

Proposition 1.32 : The extremal rays of the cone $P(X)$ are exactly the rays generated by the non zero dichotomies .

proof : From 1.25 we have only to prove that an element d of $P(X) \setminus \Pi(X)$ satisfying $d(x_1,x_2) = 1$ for some $x_1,x_2 \in X$ cannot belong to an extremal ray of $P(X)$.

Now take $d \in P(X) \setminus \Pi(X)$ and $x_1,x_2 \in X$ with $d(x_1,x_2) = 1$; since d is not a dichotomy , we can find $x_3 \in X$ such that $d(x_1,x_3) = \alpha > 0$, $d(x_2,x_3) = \beta > 0$; up to a permutation , we can assume $\alpha \geq \beta$; now , from 1.30 , there is a nonnegative measure ν on $(\Pi(X),\mathfrak{B}_X)$ such that $d = \int \pi \; \nu\,(d\pi)$ and we set $d_1 = \int \pi(x_1,x_2)\; \pi(x_1,x_3)\pi \;\; \nu(d\pi)$ and $d_2 = d-d_1$; then we have (using 1.31):

$d_1(x_1,x_2) = \frac{1+\alpha-\beta}{2} > 0$, $d_1(x_2,x_3)=0$, $d_2(x_2,x_3) = \beta > 0$; since d_1 and d_2 belong to $P(X)$, the semimetric d cannot belong to an extremal ray of $P(X)$.□

(1.33) On a finite set X , the Proposition 1.32 is implied obviously by 1.30 ; among easy consequences one can see that the convex cone $P(X)$ is closed (since it is the convex cone generated by the finite set $\Pi(X)$)

(1.34) We will see below that $P(X)$ is always closed for pointwise convergence, even if X is infinite; but it will be not so easy. Let us explain the reason of this difficulty :

(1.34.1) we will denote $S(X)$ the vector space of all applications $h: X^2 \longrightarrow \mathbb{R}$ which are symmetric (i.e. $h(x,x') = h(x',x)$ for all $x,x' \in X$) and vanish on the diagonal (i.e. $h(x,x) = 0$ for all $x \in X$) ; $S(X)$ and its subset $P(X)$ will be endowed with the topology of the pointwise convergence (i.e. the restriction of the topology of $\mathbb{R}^{X\times X}$); thus it is a complete locally convex topological vector space ;

(1.34.2) $\Pi(X)$, as a topological subspace of $S(X)$, is compact (it is implied by the criterion 1.22.3) ;

(1.34.3) nevertheless, if X is infinite, $P(X)$ is not a convex cone with compact basis (i.e. generated by a convex compact set not containing 0) : in fact, if X is infinite, $\Pi(X) \setminus \{0\}$ is not closed.

Now we are going to study the following question (for X infinite) :
let d be the limit of an ultrafilter on $P(X)$;
is d an embeddable semimetric ? more explicitely , is there a measure space $(\Omega, \mathcal{A}, \mu)$ such that (X,d) is a subspace of $K(\Omega, \mathcal{A}, \mu)$?

We will see first that the answer is yes if we allow μ to be only an additive (noonegative) measure (and not a σ-additive one as in 1.2) :
(1.35) let $\mathcal{A}$ be a Boole algebra (may be an abstract one) ; a (nonnegative) additive measure on $\mathcal{A}$ is an application $\mu : \mathcal{A} \longrightarrow [0,+\infty]$ satisfying $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$ for each finite family $(A_i)_{i \in I}$ of disjoints elements of $\mathcal{A}$;
the definition of $K(\mathcal{A},\mu)$ (definition 1.8) will be carried out without change from a σ-additive measure to an additive one .

Proposition 1.36 [4]: Let $\mathcal{U}$ be a convergent ultrafilter on $P(X)$ (resp. h-$P(X)$, and set $d_0 = \lim_{d,\mathcal{U}} d$ (i.e. $d_0(x,x') = \lim_{d,\mathcal{U}} d(x,x')$ for all $x,x' \in X$) . Then there is an algebra $\mathcal{A}$ of subsets of a set Ω and a nonnegative additive measure (resp. integer measure) μ_0 on $(\Omega, \mathcal{A})$ such that the space (X,d) is an (isometric) subspace of $K(\mathcal{A},\mu_0)$.

proof : For each $d \in P(X)$ (resp. h-$P(X)$) we fix a σ-additive nonnegative measure (resp. integer measure) on $(2^X, \mathcal{A}_X)$ such that j_X is an embedding of (X,d) into $K(2^X, \mathcal{A}_X, \mu_d)$
(j_X comes from 1.26 and the existence of such a measure comes from 1.27).
Now we set $\mu_0 = \lim_{d,\mathcal{U}} \mu_d$; clearely it exists (since $[0,+\infty]$ is compact) and it is a nonnegative additive measure (resp. integer measure) on $(2^X, \mathcal{A}_X)$; moreover we have :

$d_o(x,x') = \mu_o\ (j_X(x)\ \Delta\ j_X(x'))$ for all $x,x' \in X$. □

Now we will see that the extension (made in 1.35) of the definition of $K(\Omega,\mathcal{A},\mu)$ from a σ-additive measure to an additive one is not really an extension ; in fact the following result is known (Stone's compactification, see for example [22] p.9 , see also [4]):

Proposition 1.37 : Let $\mathcal{A}$ be a Boole algebra and let μ be a nonnegative additive measure (resp. integer measure) on $\mathcal{A}$. Then there is a σ-algebra $\hat{\mathcal{A}}$ of a set $\hat{\Omega}$, a nonnegative σ-additive measure (resp. integer mesure) $\hat{\mu}$ on $(\hat{\Omega},\hat{\mathcal{A}})$ and an (isometric) embedding of $K(\mathcal{A},\mu)$ into $K(\hat{\Omega},\hat{\mathcal{A}},\hat{\mu})$.

proof : Let use denote $\hat{\Omega}$ the set of all ultrafilters on $\mathcal{A}$; for each $A \in \mathcal{A}$ we set $\hat{A} = \{\hat{\omega} \in \hat{\Omega} \mid A \in \hat{\omega}\}$. The set $\hat{\Omega}$ will be endowed with the topology generated by the applications $\hat{\omega} \longrightarrow 1_{\hat{A}}(\hat{\omega})$; with this topology $\hat{\Omega}$ is a compact space (the Stone's representation space) and $\hat{A}$ (for each $A \in \mathcal{A}$) is a subset of $\hat{\Omega}$ which is both open and closed. Moreover the application $A \longrightarrow \hat{A}$ is an isomorphism of Boole algebras from $\mathcal{A}$ into $\mathcal{A}_o = \{\hat{A} \mid A \in \mathcal{A}\}$; hence we can define a nonnegative additive measure μ_o on $\mathcal{A}_o$ by setting $\mu_o(\hat{A}) = \mu(A)$ for each $A \in \mathcal{A}$. We observe that the algebra $\mathcal{A}_o$ is approximated by a compact class (in fact $\mathcal{A}_o$ itself which contains only compact subsets of Ω) with respect to μ_o ; thus , using the classical extension theorem (which is also true for measures with values in $[0,+\infty]$) , we can extend μ_o into a nonnegative σ-additive measure $\hat{\mu}$ on the σ-algebra $\hat{\mathcal{A}}$ generated by $\mathcal{A}_o$. If moreover we assume that μ is an integer measure, then clearely μ_o and $\hat{\mu}$ (since $\mathbb{N} \cup \{+\infty\}$ is closed in $[0,+\infty]$) are also integer measures. □

As a corollary of 1.36 and 1.37 we obtain immediately :

Proposition 1.38 : Let X be a set (finite or infinite) ; then we have :

(1.38.1) a pointwise limit (with respect to a filter) of L^1-embeddable semimetrics on X is also a L^1-embeddable semimetric ;

(1.38.2) a pointwise limit (with respect to a filter) of h-embedabble semimetrics on X is also a h-embedabble semimetric.

The statement 1.38.2 is implicit in [4] and explicitely stated in [6]. The statement 1.38.1 is due to Bretagnolle, Dacunha-Castelle, Krivine [11] p.252 : in fact it is given in [11] in the form of a result of finiteness (see below) and proved as a corollary of a result of finiteness for normed spaces (it will appear also as a particular case of some results of finiteness for some classes of kernels of positive type [4]) . The proof used here (through 1.36 and 1.37) comes from [4]

In fact the Proposition 1.38 can be written equivalently (straithforward, see for example [4]) as a result of finiteness :

Proposition 1.39 : Let d be an application from X^2 into $\mathbb{R}$.

Then d is an L^1-embedabble (resp. h-embeddable) semimetric on X if and only if $d|_Y$ is an L^1-embeddable (resp. h-embeddable) semimetric on Y for each finite subset Y of X .

proof : We fix $s \in X$; for each finite subset Y of X , we define $f_Y : X \longrightarrow Y \cup \{s\}$ by setting $f_Y(x)=x$ if $x \in Y$ and $f_Y(x) = s$ if $x \in X \setminus Y$. Thus, for each finite subset Y of X, $d_Y = (d_{|Y \cup \{s\}}) \circ f_Y$ is a L^1-embeddable (resp. h-embeddable) semimetric on X . To end, we observe that d is the limit of the semimetrics d_Y (with respect to the filtering set of all finite subsets of X) and we apply 1.38 . □

We observe that 1.39 gives some interest to the study of the finite L^1-embeddable (and h-embeddable) metric spaces (for the metric spaces of cardinality $\leqslant 5$, see Deza [15]) .

Another equivalent formulation of 1.38.1 consists to use inequalities of finite type :

(1.40) [5] The proposition 1.39 implies that a semimetric d belongs to $P(X)$ if and only if the semimetric $i,j \longrightarrow d(x_i,x_j)$ belongs to $P(\{1,\ldots,m\})$ for every choice of $m \in \mathbb{N} \setminus \{0\}$ and of $x_1,\ldots,x_m$ (not necessarily distincts) elements of X ; we will call the elements of $P'(\{1,\ldots,m\})$ (the dual cone of $P(\{1,\ldots,m\})$ inequalities valid in L^1 of order m and the elements of $S'(\{1,\ldots,m\})$ (the dual space of $S\{1,\ldots,m\}$) inequalities of order m ; each inequality of order m will be think as a real symmetric m m matrix $a = (a_{ij})_{i,j\in\{1,\ldots,m\}}$ vanishing on the diagonal.

Now, for each application $f : i \longrightarrow x_i$ of $\{1,\ldots,m\}$ into X, the transpose of the operator $d \longrightarrow d \circ f$ (from $S(X)$ onto $S(\{1,\ldots,m\})$) is an operator $a \longrightarrow a(x_1,\ldots,x_m)$ from $S'(\{1,\ldots,m\})$ into $S'(X)$ (the dual space of $S(X)$) which carries $P'(1,\ldots,m)$ into $P'(X)$ (the dual cone of $P(X)$).

More precisely we set for each $d \in S(X)$:

$$< a(x_1,\ldots,x_m),d > = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}\, d(x_i,x_j) .$$

An element d of $S(X)$ is said to satisfy the inequality a of order m if one has $< a(x_1,\ldots,x_m),d > \geqslant 0$ for each $x_1,x_2,\ldots,x_m \in X$.

Thus an equivalent formulation of 1.30 and 1.38.1 respectively is the following :

(1.40.1) an inequality of order m is valid in L^1 if and only if it is satisfied by all dichotomies ;

(1.40.2) a semimetric is L^1-embeddable if and only if it satisfies each inequality valid in L^1 .

Let us summarize the informations about $P(X)$:

Proposition 1.41 [5] $P(X)$ is the closed convex cone generated by $\Pi(X)$ (observe that, if X is infinite, $P(X)$ has no compact basis, see 1.34.3).

To end, we give a useful corollary :

Lemma 1.42 : Let $(T, \mathcal{B}, \mu)$ be a measure space $(\mu \geq 0)$ and consider an application $t \longrightarrow d_t$ of T into $P(X)$. We assume that the application $t \longrightarrow d_t(x,x')$ is measurable for each $x,x' \in X$.
Let us set $d(x,x') = \int_T d_t(x,x')\, \mu(dt)$ for each $x,x' \in X$ and suppose that it is finite for each $x,x' \in X$.

Then d is an L^1-embeddable semimetric on X.

proof : a direct verification, or the following :
d satisfies each inequality valid in L^1 ; thus, from 1.40.2, it belongs to $P(X)$. □

1.C Scale and condition of the perimeter

We begin with some observations on the cone $P(X)$:

(1.43) The cone $P(X)$ is a proper cone (a convex cone T is said to be a proper cone if one has $T \cap (-T) = \emptyset$ or $\{0\}$) : in fact $P(X)$ is included in $D(X)$ which is a proper cone.

Moreover the convex cone $P(X)$ has a non void interior (in $S(X)$) when X is a finite set ; in fact it is equivalent to show that $\Pi(X)$ contains a basis of the vectorspace $S(X)$:

LEMMA 1.44 [17] : Let X be a finite set. Then the vector space $S(X)$ (i.e. the vector space of all symmetric applications $d : X^2 \longrightarrow R$ vanishing on the diagonal) has dimension $\binom{|X|}{2}$ and the convex cone $P(X)$ is a polyhedral cone with a non empty interior in $S(X)$. Precisely let us fix $s \in X$, and consider for each subset Y of X the dichotomy π_Y associated to the partition $(Y, X \setminus Y)$ (see 1.29). Then $\{\pi_Y \mid Y \subset X \setminus \{s\}, |Y| = 1 \text{ or } 2\}$ is a basis of the vectorspace $S(X)$.

proof : The dimension of $S(X)$ is obviously $\binom{|X|}{2}$. Moreover the dichotomies π_Y (for $Y \subset X \setminus \{s\}$, $|Y| = 1$ or 2) form a basis of $S(X)$ as can be seen from the following explicit expansion :

for each $d \in S(X)$, we have :

(1.44.2) $$d = \sum_Y \lambda_Y \lambda_Y$$

with $$2\lambda_{\{x\}} = \sum_{y \in X} [d(x,y) - d(s,y)] - (|X| - 4)\, d(s,x)$$

$$2\lambda_{\{x,x'\}} = d(s,x) + d(s,x') - d(x,x')$$

and where the sum $\sum_Y$ is taken over all subsets Y of $X \setminus \{s\}$ such that $|Y| = 1$ or 2 .

Hence the convex cone $P(X)$ has a nonempty interior in $S(X)$. □

(1.45) In fact the set $\{\pi_Y \mid Y \subset X, |Y| = 2\}$ is a more symmetric basis, but the expansion would be not so simple. For a finite set X , the above Lemma means in particular that each semimetric on X (and more generally each element of $S(X)$) is the difference od two L^1-embeddable semimetrics on X .

The reason for introducing hypercubes at scale η lies in the following Lemma which will be useful in finite combinatorics :

Lemma 1.46 [3] : Let d be an integervalued semimetric on a finite set X . Then d is L^1-embeddable if and only if there is $\eta \in \mathbb{Q} \cap]0,+\infty[$ such that d is h-embeddable at scale η .

proof : The "if part" is obvious. For the "only if part" , we take d an integervalued embeddable semimetric on a finite set X and we set $N = \binom{|X|}{2}$. Using 1.33 , 1.44 and the theorem of Caratheodory (see Bourbaki [10] , chapter II , §2 , exercice 9) we see that there are N dichotomies $(\pi_i)_{i=1,\dots,N}$ on X such that d can be expressed uniquely in the form $\sum_{i=1}^{N} \alpha_i \pi_i$ with $\alpha_i \geqslant 0$ for each $i=1,\dots,N$. The π_i's and d are integervalued ; hence the α_i's are all rational numbers. Let η be a positive integer such that $\eta\alpha_i$ is an integer for each $i=1,\dots,N$. Then (using Proposition 1.30) ηd is h-embeddable. □

The above Lemma allows us to define a quantity which is important for the discrete metric spaces (and which was considered for the graphs by Blake, Gilchrist [9] :

(1.47) Let d be an integervalued L^1-embeddable semimetric on a (finite

or infinite) set X ; then the scale of (X,d) (denoted $\eta(X,d)$) is the infimum of all numbers $\eta \in]0,+\infty[$ such that (X,d) is h-embeddable at scale η
(for a finite X, it is allways finite, from 1.46 ; for an infinite X, the scale can be infinite).

An interesting but probably untractable problem would be to characterize the embeddable semimetrics on X using linear inequalities. We have seen (in 1.40) that it is sufficient to take X finite.

The problem of recognizing h-embeddable semimetrics among L^1-embeddable semimetrics is more complicated ; only the following necessary condition is known :

Proposition 1.48 [15] : Let X be a set and d an h-embeddable semimetric on X. Then d has integer values and satisfies the condition of perimeter (see 1.22.3) which we recall here :
$d(x_1,x_2) + d(x_2,x_3)+d(x_3,x_1)$ is even for each $x_1, x_2, x_3 \in X$.

proof : It comes from 1.22.3 and 1.30. Another proof consists in the following observation :

for each subsets A,B,C of a set Ω, we have :

$$1_{A\Delta B} + 1_{B\Delta C} + 1_{C\Delta A} = 2.1_Z ,$$

with $Z = (A \cup B \cup C) \setminus (A \cap B \cap C)$. □

(1.49) We come back now to the description of the realizations of a semimetricspace given in 1.13 and 1.15 . It will be interesting in some case to restrict the support of the multiplicity measure (i.e. to use only a subset of $\Pi(X)$) or equivalently to get a realization with prescribed blocks :

- this question of support of the multiplicity measure was considered for discrete metric spaces by Deza, Rosenberg [18] and, in fact, it is exactly the purpose of the block design theory but for very particular semimetrics ;

- in a more geometric setting, the criterion of Djokovic [19] for the h-embeddability of graphs and the results of Alexander [1] and Ambartzumian [2] on the plane Buffon-Sylvester problem give particularly good examples of this situation .

A very useful observation about realizations is the following :

(1.50) let (X,d) be an L^1-embeddable semimetric space and let us take $f : x \longrightarrow A_x$ a realization of (X,d) into $(\Omega,\mathfrak{A},\mu)$; let us fix a point $s \in X$, then $f^1 : x \longrightarrow A_x^1 = A_x \Delta A_s$ is also a realization of (X,d) in $(\Omega, \quad ,\mu)$ and it satisfies $A_s^1 = \emptyset$; let us call moreover ν and ν^1 the measures on $\Pi(X)$ corresponding to f and f^1 respectively (from 1.30.1) , then one has $\nu = \nu^1$. If f is a h-realization (i.e. a realization into $(\Omega,|\,|)$) then this operation appears only as the <u>choice of an origin</u> in the affine space of dimension $|\Omega|$ over $\mathbb{Z}_2$.

We end with an observation concerning the size (see 1.12) :

(1.51) Let d be a semimetric on a set X and take $\sigma \in [\,0,+\infty\,[$; we will adopt here a multiplicative notation for the element $\{-1,1\} \times X$:

for each $\varepsilon \in \{-1,1\}$ and each $x \in X$, we will write εx for (ε,x); now the <u>antipodal extension of size σ of (X,d)</u> is the space $(\{-1,1\}\times X,\tilde{d})$ where $\tilde{d}$ is defined as follows for each εx , $\varepsilon' x' \in \{-1,1\} \times X$:

$$\tilde{d}\,(\varepsilon x,\varepsilon' x') = d(x,x') \quad \text{if} \quad \varepsilon = \varepsilon'$$

$$\tilde{d}\,(\varepsilon x,\varepsilon' x') = \sigma - d(x,x') \quad \text{if} \quad \varepsilon \neq \varepsilon'$$

(generally $\tilde{d}$ is not a semimetric, but only an element of $S\,(\{-1,1\} \times X)$).

We have now the following result (a slight extension of a result of Deza [16] on elliptic patterns) :

Lemma 1.52 : The antipodal extension of size σ of a semimetric space (X,d) is a L^1-embeddable (resp. h-embeddable) semimetric space if and only if σ is the size of a realization (resp. of a h-realization) of (X,d).

proof : Let $f : x \longrightarrow A_x$ be a realization of (X,d) into $(\Omega,\mathfrak{A},\mu)$ (resp. a h-realization of (X,d) into $(\Omega,|\ |)$) of size σ ;
for each $\varepsilon \in \{-1,1\}$ and each $B \subset \Omega$ we set $\varepsilon B = B$ if $\varepsilon = +1$ and $\varepsilon B = \Omega \setminus B$ if $\varepsilon = -1$; then $\tilde{f} : \varepsilon x \longrightarrow \varepsilon A_x$ is a realization of $(\{-1,1\} \times X , \tilde{d})$ into $(\Omega,\mathfrak{A},\mu)$
(resp. a h-realization of $(\{-1,1\} \times X , \tilde{d})$ into $(\Omega,|\ |)$.
The converse is easy. $\square$

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§2 THE CONVEX CONE OF ALL L^1-INEQUALITIES

We will study here the inequalities which are valid in L^1 (or L^1-inequalities for short), i.e. the inequalities which are satisfied by all metric subspaces of L^1.

This paragraph is divided into three sections :

2.A The polygonal inequalities, 2.B Examples and counterexamples,

2.C Other extremal inequalities .

Here are some preliminary definitions :

Definition 2.1. A linear inequality of finite type is nothing but a $m{\times}m$ matrix $a = (a_{i,j})_{i,j \in \{1,\dots,m\}}$, where m is a positive integer called the order of the linear inequality.

Definition 2.2. Let X be a set and k a symmetric kernel on X .

Let $a = (a_{i,j})_{i,j \in \{1,\dots,m\}}$ be a linear inequality of finite type adapted to F . We will say that k (or (X,k)) satisfies the inequality a if we have :

$$\forall x_1,x_2,\dots,x_m \in X \ , \ \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, k(x_i,x_j) \geq 0 \ ;$$

we will say also equivalently that the inequality a is valid for k (or for (X,k)) (Note that the x_i's are not assumed to be distinct).

Moreover, let $x_1,x_2,\dots,x_m$ be elements of X , then the number $\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, k(x_i,x_j)$ is denoted $\langle a(x_1,\dots,x_m),k\rangle$

(2.3) If an inequality a is to be considered only for metric spaces (or more generally only for kernels vanishing on the diagonal), then the diagonal entries $a_{i,i}$ have no interest and will not be given .

2.A The polygonal inequalities :

We will study first the polygonal inequalities (introduced by Deza [6]), the most important example of L^1-inequalities, as will be seen in the Proposition 2.14 .

Definition 2.4. Let n be a positive integer. The (2n+1)-*polygonal inequality* (resp. the 2n-*polygonal inequality*) is the linear inequality p_{2n+1} (resp. p_{2n}) defined by :

$$p_{2n+1}(i,j) = -\frac{1}{2} \lambda_i \lambda_j$$

with $\lambda_1,\ldots,\lambda_n = -1$, $\lambda_{n+1},\ldots,\lambda_{2n+1} = 1$

(resp. $p_{2n}(i,j) = -\frac{1}{2} \lambda_i \lambda_j$

with $\lambda_1,\ldots,\lambda_n = -1$, $\lambda_{n+1},\ldots,\lambda_{2n} = 1$).

Let $x_1,\ldots,x_{2n+1}$ be points of a set X ; we will often use, instead of the notation $p_{2n+1}(x_1,\ldots,x_{2n+1})$, the "more bipartite" notation $p_{2n+1}(x_1,\ldots,x_n;x_{n+1},\ldots,x_{2n+1})$. Similarly we will often write $p_{2n}(x_1,\ldots,x_n;x_{n+1},\ldots,x_{2n})$ in place of $p_{2n}(x_1,\ldots,x_{2n})$.

Definition 2.5. Let m be an integer, $m \geq 2$. Let $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_m)$ be a m-tuple of real numbers. Then the linear inequality defined by

$$a_{ij} = -\lambda_i \lambda_j$$

is called a *negative type inequality* if we have $\sum_{i=1}^{m} \lambda_i = 0$;

it is said an *hypermetric inequality* if the *λ_i's are integers* and we have $\sum_{i=1}^{m} \lambda_i = 1$;

more generally, we will set for each $\lambda = (\lambda_1,\ldots,\lambda_m)$ in $\mathbb{R}^m$, each $x_1,\ldots,x_m$ in X and each d in S(X) :

$$\langle h\,(x_1,\ldots,x_m),d\rangle = -\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i \lambda_j\, d(x_i,x_j).$$

Notation 2.6. Let X be a set and $d : X^2 \to \mathbb{R}$ a symmetric kernel on X vanishing on the diagonal (i.e. an element of S(X)). We will say that d (and also (X,d)) is *hypermetric* (resp. of *negative type*) if d satisfies all hypermetric inequalities (resp. all inequalities of negative type). Similarly we will say that d and also (X,d) is *n-polygonal* if it satisfies the n-polygonal inequality.

We will use also an equivalent terminology for n-polygonal when n = 3,4,5,6,7,... :

triangular , quadrangular, *pentagonal* , hexagonal, heptagonal, etc... It is important to remind that these inequalities will be used *only* for elements of S(X) (for some set X) i.e. for symmetric kernels *vanishing on the diagonal*.

Before to stufy the relations between these inequalities, we will make clear the sense of the n-polygonal inequalities :

Remark 2.7. Let X be a set; take d an element of S(X) and n an integer , $n \geq 2$. Let $x_1, x_2, \ldots, x_n$ be elements of X (they are *not assumed to be distincts*). Call the (symmetric) pairs (x_i, x_j)

$(i \neq j)$ *external* if $i \leq \frac{n}{2} < j$, and *internal* otherwise.

Then d satisfies $p_n(x_1, \ldots, x_n)$ if and only if :

the sum of the values of d on the internal pairs is not larger than the sum of the values of d on the external pairs.

For small n , it gives the following

(in each figure, the broken lines ------ will join external pairs, the continuous lines ——— will join internal pairs) :

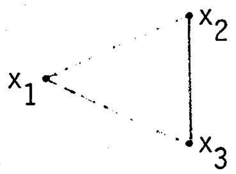
(2.7.1) 2-polygonal inequality :

$$\langle p_2(x_1;x_2),d\rangle = d(x_1,x_2)$$

(In other terms d is 2-polygonal if and only if it is nonnegative).

(2.7.2) 3-polygonal inequality (or triangular inequality)

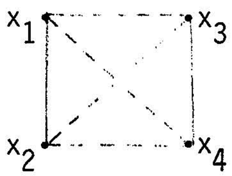
$$\langle p_3(x_1;x_2,x_3),d\rangle = d(x_1,x_2) + d(x_1,x_3) - d(x_2,x_3)$$



(In other terms d is 3-polygonal if and only if it is a semimetric)

(2.7.3) 4-polygonal inequality (or quadrangular inequality)

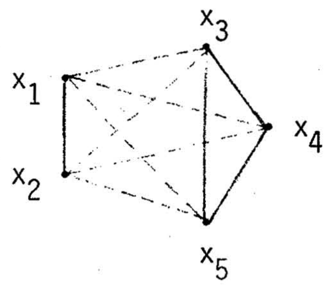
$$\langle p_4(x_1,x_2;x_3,x_4),d\rangle$$
$$= d(x_1,x_3) + d(x_1,x_4) + d(x_2,x_3) + d(x_2,x_4)$$
$$- d(x_1,x_2) - d(x_3,x_4)$$



We will see below in Proposition 2.9. that all semimetrics are quadrangular; in general a quadrangular function is only a pseudometric (see Propositions 2.9 and example 2.15.4).

(2.7.4) 5-polygonal inequality (or pentagonal inequality)

$$\langle p_5(x_1,x_2;x_3,x_4,x_5),d\rangle$$
$$= d(x_1,x_3)+d(x_1,x_4)+d(x_1,x_5)+d(x_2,x_3)+d(x_2,x_4)+d(x_2,x_5)$$
$$- d(x_1,x_2) - d(x_3,x_4) - d(x_3,x_5) - d(x_4,x_5)$$



We will see below in Proposition 2.9 that a symmetric function $d : X^2 \to \mathbb{R}$ vanishing on the diagonal and satisfying pentagonal inequality is necessary a semimetric; but there are semimetrics which are not pentagonal (see Example 2.15.6). The pentagonal inequality is certainly a very strong inequality : for example one can show (see Assouad [2]) that there are some metric spaces which cannot be Lipschitz embedded into a pentagonal metric space .

Remark 2.8. We note that an element d of $S(X)$ (i.e. a symmetric kernel $d: X^2 \to \mathbb{R}$ vanishing on the diagonal) is of negative type if and only if one has :

(2.8.1) $\forall m \in \mathbb{N} \setminus \{0\}$, $\forall\, x_1,x_2,\dots,x_m \in X$, $\forall\, \lambda_1,\lambda_2,\dots,\lambda_m \in \mathbb{R}$

$$\sum_{i=1}^{m} \lambda_i = 0 \Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \lambda_j\, d(x_i,x_j) \le 0$$

(By homogeneity the λ_i's could be taken only in the set $\mathbb{Z}$ of all integers).

Similarly an element d of S(X) is hypermetric if and only if one has :

(2.8.2) $\forall\, m \in \mathbb{N} \setminus \{0\}$, $\forall\, x_1, x_2, \ldots, x_m \in X$, $\forall\, \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{Z}$

$$\sum_{i=1}^{m} \lambda_i = 1 \Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \lambda_j d(x_i, x_j) \leq 0 .$$

Despite the striking similarity between (2.8.1) and (2.8.2) the hypermetricity is much stronger than negative type (as will be seen in Proposition 2.9 below).

The n-polygonal inequalities and the hypermetric inequalities were introduced by Deza (Tylkin) in [6] and independently rediscovered by Kelly [9] (to whom is due the terminology hypermetric).

We will show now the links between the different inequalities introduced above :

Proposition 2.9. Let X be a set and d an element of S(X) (i.e. a symmetric kernel $d: X^2 \to \mathbb{R}$ vanishing on the diagonal). Then we have :

(2.9.1) (J.B.Kelly [9]) d is hypermetric if and only if d is n-polygonal for each odd integer n ;

(2.9.2) d is of negative type if and only if d is n-polygonal for each even integer n ;

(2.9.3) d is nonnegative if and only if d is 2-polygonal;

(2.9.4) d is a semimetric if and only if d is 3-polygonal;

(2.9.5) if d is 4-polygonal, then d satisfies the following inequality :

$$\forall\, x_1, x_2, x_3 \in X \ , \ d(x_2, x_3) \leq 2[d(x_1, x_2) + d(x_1, x_3)] \ ;$$

(2.9.6) (J.B.Kelly [9]) for each integer $n \geq 4$, we have : if d is n-polygonal , then d is (n-2)-polygonal;

(2.9.7) (Deza(Tylkin) [6]) for all integers $n \geq 1$, we have if d is (2n+1)-polygonal, then d is (2n+2)-polygonal ;

(2.9.8) if d is hypermetric, then d is a semimetric of negative type.

proof : (We will use the notations p_n, t_λ , h_λ introduced in the Definitions 2.4 and 2.5).

(2.9.1) We remark first that, for each integer $n \geq 1$, one has :

$$2\, p_{2n+1} = h_\lambda$$

where $\lambda = (\lambda_1,\ldots,\lambda_{2n+1})$ with $\lambda_1 = \ldots = \lambda_n = -1$ and $\lambda_{n+1} = \ldots = \lambda_{2n+1} = 1$ (and thus $\sum_{i=1}^{2n+1} \lambda_i = 1$). In order to prove the converse, let $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_m)$ be a m-tuple of integers with $\sum_{i=1}^{m} \lambda_i = 1$; take $x_1,x_2,\ldots,x_m$ elements of X ; then $\langle h_\lambda(x_1,x_2,\ldots,x_m),d\rangle$ is nothing but $2\langle p_{2n+1}\, y_1,y_2,\ldots,y_n;y_{n+1},\ldots,y_{2n+1}),d\rangle$ where $2n+1 = \sum_{i=1}^{m} |\lambda_i|$ and where the sequence $y_{n+1},y_{n+2},\ldots,y_{2n+1}$ (resp. $y_1,y_2,\ldots,y_n$) contains, for each $j \in \{1,\ldots,m\}$ such that $\lambda_j > 0$ (resp. $\lambda_j < 0$), the element x_j of X repeated $|\lambda_j|$-times.

(2.9.2) The proof is quite similar to that of (2.9.1).

(2.9.3) and (2.9.4) It was done in Remarks 2.7.1 and 2.7.2.

(2.9.5) In order to obtain the inequality one has only to see that

$$\langle p_4(x_1,x_1;x_2,x_3),d\rangle \text{ is nonnegative.}$$

(2.9.6) Let us consider at first the case of odd n . Take $n = 2r + 1$ and let $x_1,\ldots,x_{2r-1}$, y be points of X ; we observe now that $\langle p_{2r-1}(x_1,\ldots,x_{r-1};x_r,\ldots,x_{2r-1}),d\rangle$ is nothing but $\langle p_{2r+1}(x_1,\ldots,x_{r-1},y\; ;\; y,x_r,\ldots,x_{2r-1}),d\rangle$ and it gives the result. For the case of even n , the proof is the same.

(2.9.7) Let $x_1,\dots,x_n;\ y_1,\dots,y_n$ be points of X . Then we have :

$$\langle p_{2n+2}\ x_1,\dots,x_n;y_1,\dots,y_n),d\rangle$$

$$= \frac{1}{2n-2}\Big[\sum_{i=1}^{n}\langle p_{2n+1}(x_1,\dots,\hat{x}_i,\dots,x_n;y_1,\dots,y_n),d\rangle$$

$$+\ \sum_{i=1}^{n}\langle p_{2n+1}(y_1,\dots,\hat{y}_i,\dots,y_n;x_1,\dots,x_n),d\rangle\Big]$$

(here the notation $\hat{x}_i$ means that x_i is ommitted).

(2.9.8) Assume now that d is hypermetric, then by (2.9.1) d is n-polygonal for each odd $n \geq 3$. Thus (using 2.9.7 and 2.9.6) d is n-polygonal for each even $n \geq 2$ and therefore (from 2.9.2) d is of negative type. On the other side, d is 3-polygonal and thus d is a semimetric (see 2.9.4). □

The implications between the n-polygonal inequalities can be presented by the following table :

(2.9.9)
$$\begin{array}{ccccccccc} 2 & \Leftarrow & 4 & \Leftarrow & 6 & \Leftarrow & 8 & \Leftarrow & \cdots\cdots \\ & \nearrow & & \nearrow & & \nearrow & & & \\ 3 & \Leftarrow & 5 & \Leftarrow & 7 & \Leftarrow & 9 & \Leftarrow & \cdots\cdots \end{array}$$

(where the numbers $n = 2,3,4,\dots$ correspond to the n-polygonal inequalities).

We will now give the inequalities for "covariances" which are the counterparts of the above "metric" inequalities.

We will start with some definitions :

Definition 2.10. Let m be an integer, $m \geq 2$. Let $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_m)$ be a m-tuple of real numbers. Then the linear inequality defined by :

$$a_{ij} = \lambda_i\ \lambda_j$$

is called a positive type inequality and is denoted $\tilde{t}_\lambda$.
Moreover if the λ_i's are integers the linear inequality defined by :

$$a_{ij} = \lambda_i \lambda_j \quad \text{if} \quad i \neq j$$
$$= \lambda_i^2 - \lambda_i \quad \text{if} \quad i = j$$

is denoted $\tilde{h}_\lambda$.

Let X be a set ; a symmetric kernel $k : X^2 \to \mathbb{R}$ (but not necessary vanishing on the diagonal) is called a hypermetric covariance if it satisfies $\tilde{h}_\lambda$ for all m-tuple λ of integers.

Remark 2.11. We note that the symmetric kernels $k : X^2 \to \mathbb{R}$ satisfying $\tilde{t}_\lambda$ for all m-uple λ of real numbers (or, equivalently, for all m-tuples λ of integers) are exactly the real kernels of positive type .

Proposition 2.12. Let X be a set ; take s an element of X and d an element of $S(X)$. Then we have :

(2.12.1) d is hypermetric (resp. of negative type) if and only if $K_s\, d$ is an hypermetric covariance (resp. is of positive type) (setting $K_s\, d\,(x,x') = \frac{1}{2}[d(x,s) + d(x',s) - d(x,x')]$ for each $x,x' \in X$).

(2.12.2) More precisely take $(x_0,\dots,x_n)$ a (n+1)-tuple of points of X and $\lambda = (\lambda_0,\dots,\lambda_n)$ a (n+1)-tuple of real numbers. Then we have :

$$\langle h_\lambda(x_0,x_1,\dots,x_n),d\rangle = 2\langle \tilde{h}_{\tilde{\lambda}}(x_1,\dots,x_n),K_s\, d\rangle$$

for λ_i's integers and $\sum_{i=0}^{n} \lambda_i = 1$,

and $\langle h_\lambda(x_0,x_1,\dots,x_n),d\rangle = 2\langle \tilde{t}_{\tilde{\lambda}}(x_1,\dots,x_n),K_s\, d\rangle$

for λ_i's real numbers and $\sum_{i=0}^{n} \lambda_i = 0$,

where $\tilde{\lambda} = (\lambda_1,\dots,\lambda_n)$.

proof : It is clear that one has only to prove (2.12.2).

We will give only the proof for the hypermetric case ;

assume that all λ_i's are integers and satisfy $\sum_{i=o}^{n} \lambda_i = 1$;

then a direct calculation gives :

$$(2.12.3) \qquad - \sum_{i=o}^{n} \sum_{j=o}^{n} \lambda_i \lambda_j d(x_i,x_j)$$

$$= 2[\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i\lambda_j k(x_i,x_j) - \sum_{i=1}^{n} \lambda_i k(x_i,x_i)]$$

(where we have set $k(x,x') = K_s d(x,x') = \frac{1}{2} [d(x,s) + d(x',s) - d(x,x')]$)

and it is the result. □

The next two results will show that the negative type inequalities and hypermetric inequalities are related to isometric embeddings into L^2 and L^1 respectively.

The first result is classical (Schoenberg [10]) :

Proposition 2.13. Let X be a set ; take d an element of $S(X)$.

Then d is of negative type if and only if $(X,\sqrt{d})$ is a metric subspace of a space L^2 (or equivalently of an Hilbert space).

proof : Let s be an element of X . By (2.12.1) , the kernel $K_s d$ is of positive type. Therefore there exists an application f of X into an Hilbert space H such that :

$$\forall\ x , x' \in X , \quad K_s d(x,x') = (f(x)|f(x'))_H .$$

Therefore we have for each $x,x' \in X$

$$d(x,x') = K_s d(x,x) + K_s d(x',x') - 2 K_s d(x,x')$$
$$= \|f(x) - f(x')\|_H^2 . \quad □$$

The corresponding result for l^1-spaces involves the hypermetric inequality but does not give a criterion : the hypermetric inequality appears only as a necessary condition of embedding (it can be seen [1],[4] that it is not sufficient) ; it is nevertheless sufficient for many classes of metric spaces .

Proposition 2.14. (Deza [6] in the discrete case , Kelly [9] in the general case).
Let X be a set and d a semimetric on it ; assume that d is (isometrically) embedable into L^1 ; then d is hypermetric.

proof : Let s be a point of X . Using Proposition 2.12 we have only to prove that $K_s d$ is an hypermetric covariance; by 1.50 one can choose a realization $x \longrightarrow A_x$ of (X,d) with $A_s = \emptyset$; thus there are $(\Omega, \mathcal{A})$ a measurable space , μ a nonnegative measure on it and $x \to A(x)$ an application of X into $\mathcal{A}$ such that :

$$\forall x,x' \in X \ , \ K_s\, d(x,x') = \mu(A(x) \cap A(x')) \ .$$

Let $x_1,x_2,\ldots,x_n$ be elements of X and $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ a n-uple of integers. Then we have :

$$\langle \tilde{h}_\lambda(x_1,\ldots,x_n) \ , \ K_s\, d \rangle = \int_\Omega [N^2(\omega) - N(\omega)]\mu(d\omega)$$

with $\quad N(\omega) = \sum_{i=1}^{n} \lambda_i \, 1_{A(x_i)}(\omega) \quad ;$

the function N is integer-valued , thus $N^2 - N$ is nonnegative ans it gives the result. □

Not that if we use in the proof $N^2 \geq 0$ instead of $N^2 - N \geq 0$, then we obtain only that d is of negative type.

Now we will see that the converse of the implications (2.9.6) , (2.9.7) and (2.9.8) are not true ; it will be done in the next section .

2.B Examples and counterexamples :

Examples 2.15.

example (2.15.1)

Let d be the kernel on $\mathbb{R}$ defined as follows :

$$d(s,t) = (s-t)^2 \text{ for each } s,t \in \mathbb{R}$$

Then d is of negative type, but d is not a semimetric (i.e. do not satisfy the triangular inequality).

proof : It is well known that :

- the function d is of negative type (using Proposition 2.13) since it is the square of the euclidean metric on $\mathbb{R}$;
- on the other side d is not a semimetric; one has for example :

$$d(0,2) = 4 > 2 = d(0,1) + d(1,2). \quad \square$$

example (2.15.2) [3]

Let X be a set with $|X| = 2N$; let $X = X_1 \cup X_2$ be a partition of X into two subsets having N elements each. We will define an element d of S(X) as follows :

$$\begin{aligned} d(x,x') &= 0 \text{ if } x = x' \\ &= 1 \text{ if } x \in X_1, x' \in X_2 \\ &= a \text{ otherwise.} \end{aligned}$$

Let n be an integer (with $n+2 \le N$). Then the following assertions are equivalent :

(i) d is (2n+1)-polygonal ,

(ii) d is (2n+2)-polygonal ,

(iii) $0 \le a \le 1 + \frac{1}{n}$.

proof : For $a \in [0,1]$, it is easy to see that d is embeddable into L^1. For $a > 1$, one can check that d is (2n+1)-polygonal if

$\langle p_{2n+1}(x_1,\dots,x_n;x_{n+1},\dots,x_{2n+1}),d\rangle$ is nonnegative for each distinct elements $x_1,\dots,x_n$ of X_1 and each distinct elements $x_{n+1},\dots,x_{2n+1}$ of X_2 ;
the same is true for (2n+2)-polygonal and it gives the result. □

The two above examples are sufficient to see that all the implications of the table 2.9.9 are irreversible .

Now we will give other illustrations.

example (2.15.3) [3]

Set $X = \{1,2,3\}$. We define an element d of $S(X)$ as follows :

$$d(1,2) = 2 \quad , \quad d(1,3) = \frac{1}{2} + a \quad , \quad d(2,3) = \frac{1}{2} - a$$

and naturally, $d(i,i)= 0$ for each $i = 1,2,3$.

Let n be an integer; then the following assertions are equivalent:

(i) d is (4n+2)-polygonal,

(ii) d is (4n+4)-polygonal,

(iii) $|a| \le \frac{1}{4n+1}$.

proof : an easy calculation. □

It is known that (see Deza [6] and also §3 below) :

- for $|X| \le 4$, a 3-polygonal function is necessary hypermetric;
- for $|X| = 5$, a 5-polygonal function is necessary hypermetric (and, in fact, it is embeddable into L^1 in each case).

The above example (2.15.3) shows that, even for $|X| = 3$, none of the even n-polygonal inequalities implies negative type.

example (2.15.4) [3]

Set $X =\{1,2,3\}$. We define an element d of $S(X)$ as follows :

$$d(1,2) = 2 \quad , \quad d(1,3) = 1 \quad , \quad d(2,3) = 0$$

and, naturally $d(i,i) = 0$ for each $i = 1,2,3$

(It is exactly example 2.15.3 for $a = \frac{1}{2}$).

Then d is 4-polygonal but not 3-polygonal.

Moreover, the only extremal rays of the convex xone of all 4-polygonal functions on $X =\{1,2,3\}$ correspond precisely to d and all its permutations.

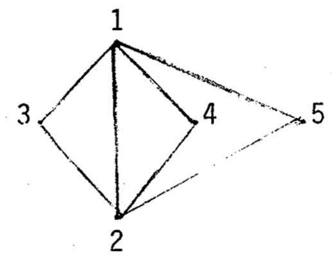
proof : easy. □

Each of the four next examples will consist of a graph $G = (X,E)$. More precisely in each case we will consider the set X of vertices of G endowed with the following metric d_G, called the truncated metric: for each distinct vertices $x,x' \in X$, set $d_G(x,x') = 1$ if x and x' are adjacent (i.e. if $(x,x') \in E$) and $d_G(x,x') = 2$ otherwise.

Note that the truncated metric will coincide here with the pathmetric (except for 2.15.8).

example (2.15.5)

Let G be given by the following figure :



Then d_G is of negative type, but d_G is not 5-polygonal.

<u>proof</u> : We remark that the application :

$$\text{(i)} \quad \begin{cases} 1 \longrightarrow 0 & \\ 2 \longrightarrow \frac{1}{2}\left(\sum_{i=1}^{8} e_i\right) , & 4 \longrightarrow e_3 + e_4 \\ & 5 \longrightarrow e_5 + e_6 \\ 3 \longrightarrow e_1 + e_2 & \end{cases}$$

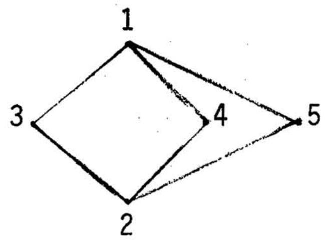
of X into $\mathbb{R}^8$ (with its orthonormal basis $(e_i)_{i \in \{1,\dots,8\}}$) is an isometric embedding of $(X, \sqrt{2}\, d_G)$ into $\mathbb{R}^8$ with the euclidean metric. Thus (using Proposition 2.13) d_G is of negative type.
On the other side $\langle p_5(1,2;3,4,5), d_G \rangle$ is negative. □

On the contrary a normed space with a norm of negative type is always embeddable into L^1 (see Bretagnolle, Dacunha-Castelle, Krivine [5]).

Note also that the application (i) describes the graph $K_{1,3}$ as a subgraph of the root system E_8 .

<u>example</u> (2.15.6)

Let G be given by the following figure :



(i.e. it is the complete bipartite graph $K_{2,3}$)

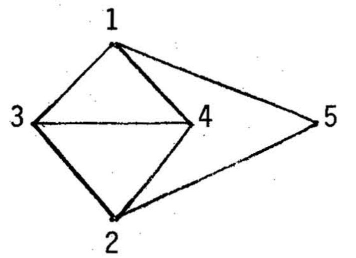
Then d_G is not 6-polygonal (and thus not 5-polygonal).

<u>proof</u> : We have only to check that

$$\langle p_6(1,1,2;3,4,5), d_G \rangle \text{ is negative .} \quad \square$$

example (2.15.7)

Let G be given by the following figure



Then d_G is neither 5-polygonal nor 8-polygonal.

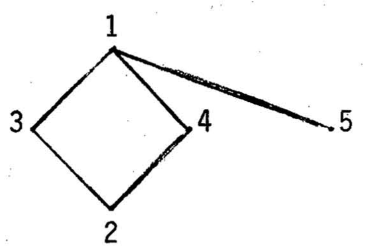
proof : We have only to check that

$\langle p_5(1,2;3,4,5), d_G \rangle$ and $\langle p_8(1,1,2,2;3,4,5,5), d_G \rangle$

are negative. □

example (2.15.8)

Let G be given by the following figure



Then d_G is neither 5-polygonal nor 10-polygonal.

proof : We have only to check that

$\langle p_5(1,2;3,4,5), d_G \rangle$

and $\langle p_{10}(1,1,1,2,2;3,3,4,4,5), d_G \rangle$ are negative. □

(2.16) In fact, one can see that the graph $K_{2,3}$ and its permutations correspond to the only non embeddable extremal rays of the convex cone of all semimetrics on $X = \{1,2,3,4,5\}$. Thus the example (2.15.6) is the "fundamental" example of a non embeddable semimetric (the examples 2.15.5, 2.15.7 and 2.15.8 consist only to add or to drop an edge to $K_{2,3}$) .

2.C Other extremal inequalities :

A linear inequality of order m is called valid in L^1 if it is satisfied by each metric subspace of L^1 (see 1.40) ; it is called extremal if moreover it belongs to an extremal ray of the convex cone of all inequalities valid in L^1 of order m .

For $m \leq 5$, each extremal inequality is (see [6]) a multiple of a polygonal inequality .

For $m = 6$, the same result is probably true .

But, for $m = 7$, the situation changes and one can give (Assouad [1]) a linear inequality of order 7 which is extremal and is not a multiple of a polygonal inequality :

Proposition 2.17 (Assouad [1]). Let (X,d) be a metric space Suppose that (X,d) is embeddable into L^1 , then we have for each $x_1, x_2, \ldots, x_7$ elements of X :

(2.17.1) $$5\, d(x_1,x_2) + 2 \sum_{i=4}^{7} d(x_3,x_i) + \sum_{i=4}^{7} \sum_{j=4}^{i=1} d(x_i,x_j)$$
$$\leq 5\, d(x_1,x_3) + 3 \sum_{i=4}^{7} d(x_1,x_i) + 3\, d(x_2,x_3) + 2 \sum_{i=4}^{7} d(x_2,x_i).$$

On the other side hypermetric metric spaces do not necessarily satisfy the inequality (2.17.1).

proof : See [1] . □

The dual (and independently found, Avis [4]) form of this result is that there are some hypermetric spaces (X,d) with X = 7 which are not embeddable into L^1 .

An interesting problem consists to determine all extremal hypermetric inequalities on a given finite set (see Deza [8]) .

An inequality is valid in L^1 if and only if it is satisfied by all dichotomies (see 1.32) ; hence we obtain :

Proposition 2.18 (Assouad [3]) Let $a = (a_{ij})_{i,j \in \{1,\dots,m\}}$ be an (m,m) matrix which is real valued and symmetric. Let $(\varepsilon_i)_{i \in \{1,\dots,m\}}$ be a sequence of -1 and +1 such that one has for each other sequence $(\eta_i)_{i \in \{1,\dots,m\}}$ of -1 and +1 :

(2.18.1) $$\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, \varepsilon_i\, \varepsilon_j \geq \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, \eta_i\, \eta_j$$

Then the matrix $(a_{ij}\, \varepsilon_i\, \varepsilon_j)_{i,j \in \{1,\dots,m\}}$ is an L^1-inequality. In other terms we have (for each metric subspace (X,d) of L^1 and each m-uple $(x_1,\dots,x_m)$ of elements of X) :

(2.18.2) $$\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, \varepsilon_i\, \varepsilon_j\, d(x_i,x_j) \geq 0 .$$

Conversely (2.18.2) implies (2.18.1).

Proof : Let $\pi: x,x' \to |1_A(x) - 1_A(x')|$ be a dichotomy and set $\eta(x) = 2\, 1_A(x) - 1$. We have for each $x,x' \in X$:

$$|\eta(x)| = |\eta(x')| = 1 \text{ and } \pi(x,x') = \left|\frac{\eta(x) - \eta(x')}{2}\right| = \left(\frac{\eta(x)-\eta(x')}{2}\right)^2 = \tfrac{1}{2}(1-\eta(x)\eta(x')).$$

Thus we have for each $x_1,\dots,x_m$ in X :

$$2 \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\varepsilon_i\varepsilon_j\, d(x_i,x_j) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\varepsilon_i\varepsilon_j - \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\varepsilon_i\eta(x_i)\varepsilon_j\eta(x_j).$$

The gives the result. □

Moreover, if the quadratic form $\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, \eta_i\, \eta_j$ attains its maximal value at two different places, then the corresponding inequalities are said <u>switching equivalent</u> together (this operation appears in [6], but the above presentation, linked with switching of graphs, is due to [3]).

2.19 A list of extremal inequalities valid in L^1 :

We list here some inequalities valid on L^1 and of order 7 . We will use the following notation :

the inequality $a = (a_{ij})_{i,j\in\{0,\dots,6\}}$ will be written

$$a = (a_{01},a_{02},a_{03},a_{04},a_{05},a_{06};a_{12},a_{13},a_{14},a_{15},a_{16};$$
$$a_{23},a_{24},a_{25},a_{26};a_{34},a_{35},a_{36};a_{45},a_{46};a_{56}).$$

Now we are going to give our list (due to [3]) :

$$a^1 = (-1,\ 1,\ 1,\ 1,\ 1,\ 1;\ 1,\ 1,\ 1,\ 1,\ 1;\ 0,-1,-1,\ 0;\ 0,-1,-1;\ 0,-1;\ 0\ ),$$
$$a^2 = (-1,-1,\ 0,\ 0,\ 1,\ 1;-1,\ 1,\ 1,\ 1,\ 1;\ 1,\ 1,\ 1,\ 1;-1,-1,\ 0;\ 0,-1;\ 0\ ),$$
$$a^3 = (\ 1,\ 0,\ 1,\ 0,\ 1,\ 1;\ 1,-1,\ 0,\ 0,-1;\ 1,-1,\ 0,\ 1;\ 1,-1,-1;\ 1,\ 1;-1\ ),$$
$$a^4 = (-1,-1,-1,\ 1,\ 1,\ 1;-1,\ 0,\ 0,\ 1,\ 1;\ 0,\ 1,\ 0,\ 1;\ 1,\ 1,\ 1;\ 0,-1;-1\ ),$$
$$a^5 = (\ 0,\ 1,\ 1,\ 1,\ 1,\ 0;-1,\ 0,\ 1,\ 1,\ 1;-1,\ 0,\ 0,\ 1;-1,\ 0,\ 1;-1,\ 0;-1\ ),$$
$$a^6 = (-1,-1,\ 0,\ 0,\ 1,\ 1;\ 0,-1,\ 1,\ 1,\ 0;\ 1,-1,\ 0,\ 1;\ 1,\ 1,\ 0;\ 0,\ 1;-1\ ),$$
$$a^7 = (-1,\ 1,\ 0,\ 1,\ 0,\ 1;\ 1,\ 1,\ 0,\ 0,\ 1;-1,\ 0,\ 1,\ 0;\ 1,\ 1,\ 0;-1,-1,-1\ ),$$
$$a^8 = (-1,-1,\ 1,\ 1,\ 2,\ 2;-1,\ 1,\ 1,\ 2,\ 2;\ 1,\ 1,\ 2,\ 2;\ 0,-2,-1;-1,-2;-3\ ),$$
$$a^9 = (-3,-3,-3,-3,-5,\ 5;\ 1,\ 1,\ 1,\ 2,-2;\ 1,\ 1,\ 2,-2;\ 1,\ 2,-2;\ 2,-2;-3\ ),$$
$$a^{10} = (-1,-1,-1,\ 2,\ 2,\ 3;-1,-1,\ 2,\ 2,\ 3;-1,\ 2,\ 2,\ 3;\ 2,\ 2,\ 3;-3,-5;-5\ ).$$

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